A moduli space of character sheaves

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Abstract. We study de Rham character sheaves on a commutative connected algebraic group G, defined as multiplicative line bundles with integrable connection. We construct a group algebraic space $G^\flat$ representing their moduli problem on seminormal test schemes, and we investigate its functoriality and geometry. The main technical ingredient is a study of extension sheaves on the de Rham space G_{dR}. An appendix provides self-contained, elementary proofs of basic results on de Rham spaces that may be of independent interest.

1. Introduction

Motivation from number theory

Let p and ℓ be distinct prime numbers, and let G be a commutative connected algebraic group over $\mathbb{F}_p$. Denoting by $\mathrm{Fr}_G \colon G \to G$ the arithmetic Frobenius, the *Lang isogeny*

$$L_G \colon G \to G, \qquad x \mapsto \mathrm{Fr}_G(x)\, x^{-1}.$$

is a finite étale Galois covering with automorphism group $G(\mathbb{F}_p)$. Consequently, L_G realizes $G(\mathbb{F}_p)$ as a quotient of $\pi_1^{\text{ét}}(G)$. A character $\chi \colon G(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^\times$ then naturally induces a rank-one ℓ-adic representation of $\pi_1^{\text{ét}}(G)$:

$$\pi_1^{\text{ét}}(G) \twoheadrightarrow G(\mathbb{F}_p) \xrightarrow{\chi^{-1}} \overline{\mathbb{Q}}_\ell^\times .$$

Equivalently, we obtain a rank-one ℓ-adic local system on G that we denote by $\mathscr{L}_\chi$.

Write $m \colon G \times G \to G$ for the group law. Among rank-one ℓ-adic local systems on G, the local systems above are precisely the *multiplicative* ones; those admitting an isomorphism

$$m^*\mathscr{L} \simeq \mathscr{L} \boxtimes \mathscr{L},$$

reflecting the identity $\chi(xy) = \chi(x)\chi(y)$. More precisely, the assignment

$$\left\{ \begin{matrix} \text{Characters} \\ \chi \colon G(\mathbb{F}_p) \to \overline{\mathbb{Q}}_\ell^\times \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Isomorphism classes of rank-one } \ell\text{-adic local} \\ \text{systems } \mathscr{L} \text{ on } G \text{ with } m^*\mathscr{L} \simeq \mathscr{L} \boxtimes \mathscr{L} \end{matrix} \right\}$$

is an isomorphism of groups [ST21, Lem. 2.16].

This geometrization of characters has had far-reaching consequences in number theory. It underlies Deligne's ℓ-adic Fourier transform, which Laumon used to give a striking simplification of Deligne's proof of Weil II [Del80; Lau87]. Since then, Deligne's theory of weights has remained a central input in analytic number theory through trace functions and character sums; see [Del71; KL85; KMS17; FFK25] for some remarkable examples.

Character sheaves

Henceforth, we assume that the base field k has characteristic zero. Motivated by the discussion above, we introduce the following *de Rham* counterpart of the multiplicative local systems $\mathscr{L}_\chi$.

Definition. Let S be a k-scheme. A *character sheaf on* G *relative to* S is a line bundle $\mathscr{L}$ on G_S equipped with an integrable connection ∇ relative to S satisfying

$$m_S^*(\mathscr{L}, \nabla) \simeq (\mathscr{L}, \nabla) \boxtimes (\mathscr{L}, \nabla).$$

In the absolute case $S = \operatorname{Spec} k$, we call such an object a *character sheaf on* G.

For reasons that will soon become apparent, we denote by $H^1_m(G_{dR} \times S, \mathbb{G}_m)$ the group of isomorphism classes of character sheaves on G relative to S. To get a first grip on these objects, we recall the Barsotti–Chevalley theorem: the algebraic group G fits into a short exact sequence

$$0 \to T \times U \to G \to A \to 0,$$

where T is a torus, U is a unipotent group, and A is an abelian variety. The groups $H^1_m(G_{dR}, \mathbb{G}_m)$ have straightforward descriptions for these building blocks.

■ Example. In characteristic zero, a unipotent group U is necessarily a vector group. The group $H^1_m(U_{dR}, \mathbb{G}_m)$ then identifies with the dual vector space U^*. Choosing coordinates $x_1, \ldots, x_n$ on U, the isomorphism is given by

$$\begin{aligned} k^n &\to H^1_m(U_{dR}, \mathbb{G}_m) \\ (\chi_1, \ldots, \chi_n) &\mapsto (\mathcal{O}_U, d - \chi_1\, dx_1 - \cdots - \chi_n\, dx_n)\,. \end{aligned}$$

Given a torus T with character group X and Lie algebra $\mathfrak{t}$, the group $H^1_m(T_{dR}, \mathbb{G}_m)$ is isomorphic to $\mathfrak{t}^*/X$. After a finite extension of k, we may assume that T is split with coordinates $t_1, \ldots, t_r$. The isomorphism is then explicitly given by

$$\begin{aligned} (k/\mathbb{Z})^r &\to H^1_m(T_{dR}, \mathbb{G}_m) \\ (\chi_1, \ldots, \chi_r) &\mapsto \left(\mathcal{O}_T, d - \chi_1 \frac{dt_1}{t_1} - \cdots - \chi_r \frac{dt_r}{t_r}\right). \end{aligned}$$

For abelian varieties A, every line bundle with integrable connection is a character sheaf. This is because a line bundle equipped with an integrable connection has a vanishing first Chern class, implying that it lies in $\mathrm{Pic}^0(A)$. (See [Lau96, Lem. 2.1.1].) ∎

A representability theorem

In [MM74], Mazur and Messing constructed a moduli space of line bundles with integrable connection on an abelian variety. By the discussion above, this is equivalently a moduli space of character sheaves. This construction may be viewed as an early instance of the moduli spaces that later became central in non-abelian Hodge theory [Sim94]. For affine groups, however, the collection of line bundles with flat connection is typically far too large to be algebro-geometric in any reasonable sense.[1] Imposing the character condition restores finiteness and leads to a well-behaved moduli problem.

Theorem A (3.17). *There exists a smooth commutative connected group algebraic space* $G^\flat$ *satisfying* $\dim G \leqslant \dim G^\flat \leqslant 2\dim G$, *whose* S*-points parametrize character sheaves on* G *relative to* S *for every seminormal* k*-scheme* S.

Sketch of the proof. The inspiration for this construction is the Barsotti–Weil formula [RR25, Cor. 3.6]. For an abelian variety A, this formula asserts that the extension sheaf $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$, computed in the category of abelian sheaves on the fppf site $(\mathsf{Sch}/k)_{\mathrm{fppf}}$, is representable by the dual abelian variety A'. This is a moduli space of line bundles $\mathscr{L}$ on A satisfying $m^*\mathscr{L} \simeq \mathscr{L} \boxtimes \mathscr{L}$.

The next ingredient is the so-called *de Rham space*. For a smooth algebraic variety X, line bundles on the de Rham space X_{dR} correspond to line bundles with flat connection on X. Consequently, it can be shown that the abelian sheaf $A^\natural := \underline{\mathrm{Ext}}^1(A_{\mathrm{dR}}, \mathbb{G}_m)$ parametrizes character sheaves on A and is representable by a commutative connected algebraic group. This provides a simpler construction of the Mazur–Messing algebraic group.

For a general commutative connected algebraic group G and a reduced k-scheme S, Theorem 2.28 provides a functorial isomorphism between the S-points of the sheaf $G^\natural := \underline{\mathrm{Ext}}^1(G_{\mathrm{dR}}, \mathbb{G}_m)$ and the group $H^1_m(G_{\mathrm{dR}} \times S, \mathbb{G}_m)$. If G is semiabelian (i.e., an extension of an abelian variety by a torus), then this abelian sheaf is representable by a commutative connected group algebraic space, thereby establishing the theorem.

However, when G contains a unipotent subgroup U, the sheaf $G^\natural$ is no longer representable by an algebraic space. Proposition 3.10 shows that the issue comes from a quotient of $G^\natural$ isomorphic to the abelian sheaf $\underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$, studied in [RR25], which is nonzero but vanishes on all seminormal k-schemes. Discarding this quotient produces the desired moduli space $G^\flat$. □

[1] For instance, the "moduli space" of line bundles with integrable connection on $\mathbb{A}^1$ is the ind-scheme $\mathbb{A}^\infty = \mathrm{colim}_n\, \mathbb{A}^n$, whereas the moduli space of character sheaves on $\mathbb{G}_a$ is $\mathbb{G}_a$ itself.

For the reader's convenience, we record that every smooth k-scheme is seminormal. The need for this regularity hypothesis stems from the fact that G need not be proper. By contrast, when G is an abelian variety, the S-points of $G^\flat$ parametrize character sheaves on G relative to S for *every* k-scheme S. A similar phenomenon occurs in [Ros25], where the author proves the (almost-)representability of the Picard functor for non-proper schemes after restricting to smooth test schemes.

Functorial and geometric properties of the moduli space

Motivated by the analogy between the moduli space $G^\flat$ and the Pontryagin dual of a locally compact abelian group, one might expect that a short exact sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

of commutative connected algebraic groups induces a short exact sequence

$$0 \to G_3^\flat \to G_2^\flat \to G_1^\flat \to 0 \tag{$*$}$$

of abelian fppf sheaves. Somewhat surprisingly, this fails in general; see Example 3.21 for a counterexample in which G_1 is an elliptic curve and $G_3 = \mathbb{G}_m$. Nevertheless, the following theorem shows that ($*$) is exact in many cases.

Theorem B (3.16, 3.18, 3.19). *The sequence ($*$) is always left exact and right exact. Moreover, it is exact in the middle in each of the following cases:*

1. G_1 *is linear and* G_3 *is an abelian variety;*
2. G_1, G_2, G_3 *are all linear;*
3. G_1, G_2, G_3 *are all abelian varieties.*

We next turn to the geometry of these moduli spaces. For an abelian variety A, certain *big* subsets of $A^\flat$ play a central role in non-abelian Hodge theory [Sim93] and in the formulation of generic vanishing theorems for de Rham cohomology [Sch15]. For a torus T, similar *big* subsets of $T^\flat$ were studied in [Sab92]. Here, we extend the definition of these subsets to arbitrary commutative connected algebraic groups.

Definition (3.22, 3.25). For an epimorphism $\rho\colon G \twoheadrightarrow \widetilde{G}$ with connected kernel, the image of $\rho^\flat\colon \widetilde{G}^\flat \hookrightarrow G^\flat$ is said to be a *linear subspace* of $G^\flat$. A *generic subspace* of $G^\flat$ is the complement of a finite union of translates of linear subspaces of $G^\flat$ with positive codimension.

In the affine case these notions reduce to familiar linear-algebraic ones, while for abelian varieties they recover the subsets studied by Simpson and Schnell.

■ Example — 3.23. Let T be a torus with character group X and Lie algebra $\mathfrak{t}$. A linear subspace of $\mathrm{T}^\flat \simeq \mathfrak{t}^*/\mathrm{X}$ is of the form V/Y, where Y is a subgroup of X and V is the linear subspace of $\mathfrak{t}^*$ generated by Y, in the sense of linear algebra. For a unipotent group U, a linear subspace of $\mathrm{U}^\flat \simeq \mathrm{U}^*$ corresponds to a linear subspace of the underlying vector space of U^*, in the sense of linear algebra.

For an abelian variety A, linear subspaces of $\mathrm{A}^\flat \simeq \mathrm{A}^\natural$ were first studied by Simpson, who termed them *triple tori* [Sim93, p. 365]. Schnell refers to translates of linear subspaces of $\mathrm{A}^\flat$ as *linear subvarieties* [Sch15, Def. 2.3]. ■

We refer to Subsection 3.3 for a detailed study of linear and generic subspaces. For example, when $k = \mathbb{C}$, we consider the analytification $\mathrm{G}^\flat_{\mathrm{an}}$ of the moduli space $\mathrm{G}^\flat$, and we show that if V is a generic subspace of $\mathrm{G}^\flat$, then $\mathrm{V}_{\mathrm{an}} \subset \mathrm{G}^\flat_{\mathrm{an}}$ is open and dense with respect to the analytic topology; see Proposition 3.28.

Comparison with Betti character sheaves

Replacing the coefficients of de Rham cohomology by their Betti counterparts yields another natural geometrization of characters: rank-one local systems of $\mathbb{C}$-vector spaces on the analytification G_{an}.[2] Equivalently, they correspond to characters of the topological fundamental group $\pi_1(\mathrm{G}_{\mathrm{an}}) \to \mathbb{C}^\times$.

These objects have a well-studied moduli space in the literature: the character variety

$$\mathrm{Char}(\mathrm{G}) := \mathrm{Spec}(\mathbb{C}[\pi_1(\mathrm{G}_{\mathrm{an}})]).$$

Its $\mathbb{C}$-points identify with characters $\pi_1(\mathrm{G}_{\mathrm{an}}) \to \mathbb{C}^\times$, hence with rank-one local systems on G_{an}. Moreover, the Hopf algebra structure on $\mathbb{C}[\pi_1(\mathrm{G}_{\mathrm{an}})]$ endows $\mathrm{Char}(\mathrm{G})$ with a natural structure of algebraic group, corresponding to tensor product of local systems.

On the de Rham side, a character sheaf $(\mathscr{L}, \nabla)$ on G determines a rank-one local system on G_{an} by taking horizontal sections. This construction induces a holomorphic morphism of groups

$$\mathrm{G}^\flat_{\mathrm{an}} \to \mathrm{Char}(\mathrm{G})_{\mathrm{an}}.$$

Our final theorem, a consequence of the Riemann–Hilbert correspondence, identifies these analytic moduli spaces for semiabelian varieties.

Theorem C (3.30). *The Riemann–Hilbert map $\mathrm{G}^\flat_{\mathrm{an}} \to \mathrm{Char}(\mathrm{G})_{\mathrm{an}}$ is surjective. Moreover, it is an isomorphism if and only if G is a semiabelian variety.*

For unipotent groups, the Riemann–Hilbert map is far from being an isomorphism: their character varieties are trivial, whereas their nontrivial character sheaves are the abundant exponential connections. These are the prototypical examples of irregular

[2] In this setting, multiplicativity is automatic, so these are precisely the multiplicative local systems on G_{an}.

connections. Accessing them on the Betti side would require an enhancement of the category of local systems, such as the irregular constructible sheaves of Kuwagaki [Kuw21] or the wild Betti sheaves of Scholze [Sch25]. We leave the development of this direction to future work.

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This paper is dedicated to Gérard Laumon. This project grew out of an attempt to understand his paper [Lau96]; after kindly listening to my questions, he offered characteristically lucid advice on how to approach them. As a naive doctoral student, I promptly tried a different route—no doubt with Laumon's ideas firmly at work in the background—and most of the results of this paper were proved the following week.

2. Cartier duality of commutative group stacks

2.1. Definitions and first properties

Let Lat be the full subcategory of Ab whose objects are free abelian groups of finite rank. Given an ∞-category C with finite products, we define its category of *abelian group objects* $\mathsf{Ab}(\mathsf{C})$ as the ∞-category of functors $\mathsf{Lat}^{\mathrm{op}} \to \mathsf{C}$ that commute with finite products. (See [Lur17, §1.2] for more.)

When C is the category of sets, $\mathsf{Ab}(\mathsf{C})$ coincides with the usual category of abelian groups. If C is the ∞-category of anima[3], $\mathsf{Ab}(\mathsf{C})$ is the ∞-category $\mathsf{An}(\mathsf{Ab})$ of animated abelian groups which, by the Dold–Kan correspondence, is equivalent to $\mathsf{D}^{\leqslant 0}(\mathsf{Ab})$. More generally, if X is an ∞-topos, then $\mathsf{Ab}(\mathsf{X})$ is the ∞-category of sheaves on X valued in animated abelian groups. This motivates the definition below.

Definition 2.1 (Commutative group stack). Let C be a 1-site. A *commutative group stack* on C is a sheaf

$$\mathsf{C}^{\mathrm{op}} \to \mathsf{An}_{\leqslant 1}(\mathsf{Ab}),$$

where $\mathsf{An}_{\leqslant 1}(\mathsf{Ab})$ is the full subcategory of $\mathsf{An}(\mathsf{Ab})$ consisting of animated abelian groups M with $\pi_i(M) = 0$ for $i \neq 0, 1$.

Equivalently, a commutative group stack on C is an abelian group object in the 2-topos $\mathsf{Sh}(\mathsf{C}, \mathsf{Grpd})$. These objects were first suggested by Grothendieck and subsequently

[3] We refer to [ČS24, §5.1] for the definition of animation and of the ∞-category An of anima (also known as the ∞-category of spaces or ∞-groupoids).

studied by Deligne in [SGA 4_{III}, Exp. XVIII, §1.4] under the name *champs de Picard strictement commutatifs*.

■ Remark 2.2. The forgetful functor $\mathsf{An}(\mathsf{Ab}) \to \mathsf{An}$ restricts to $\mathsf{An}_{\leqslant 1}(\mathsf{Ab}) \to \mathsf{Grpd}$. Concretely, an object of $\mathsf{An}_{\leqslant 1}(\mathsf{Ab})$ is a groupoid M endowed with a symmetric monoidal structure $+$ such that the set of isomorphism classes $\pi_0(M)$ is a group. Additionally, for all x, y in M, the symmetry constraints $x + x \to x + x$ and $x + y \to y + x \to x + y$ are supposed to be the identity maps. ■

The following proposition, a consequence of the Dold–Kan correspondence, provides a concrete characterization of commutative group stacks. This characterization will underpin virtually every computation involving these objects throughout this paper.

Proposition 2.3 (Deligne). *Let C be a 1-site. Denote by $\mathsf{Ch}^{[-1,0]}(\mathsf{Ab}(\mathsf{C}))$ the category of complexes of abelian sheaves*[4] *on C concentrated in degrees -1 and 0. Similarly, denote by $\mathsf{D}^{[-1,0]}(\mathsf{Ab}(\mathsf{C}))$ the full subcategory of $\mathsf{D}(\mathsf{Ab}(\mathsf{C}))$ consisting of the objects whose cohomologies are concentrated in degrees -1 and 0. The functor*

$$\begin{aligned} \mathsf{Ch}^{[-1,0]}(\mathsf{Ab}(\mathsf{C})) &\to \mathsf{Sh}(\mathsf{C}, \mathsf{An}_{\leqslant 1}(\mathsf{Ab})) \\ [\mathscr{F} \to \mathscr{G}] &\mapsto [\mathscr{G}/\mathscr{F}] \end{aligned}$$

factors through $\mathsf{D}^{[-1,0]}(\mathsf{Ab}(\mathsf{C}))$, and the induced functor $\mathsf{D}^{[-1,0]}(\mathsf{Ab}(\mathsf{C})) \to \mathsf{Sh}(\mathsf{C}, \mathsf{An}_{\leqslant 1}(\mathsf{Ab}))$ is an equivalence of ∞-categories.

Proof. Let D be an ∞-category that admits small limits. Recall that a functor $\mathscr{F} : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ is a sheaf if it preserves finite products and if, for every covering $X \to S$ in C, the natural morphism

$$\mathscr{F}(S) \to \lim \left[\mathscr{F}(X) \rightrightarrows \mathscr{F}(X \times_S X) \substack{\textstyle\to\\[-0.3em]\textstyle\to\\[-0.3em]\textstyle\to} \mathscr{F}(X \times_S X \times_S X) \substack{\textstyle\to\\[-0.3em]\textstyle\to\\[-0.3em]\textstyle\to\\[-0.3em]\textstyle\to} \cdots \right]$$

is invertible. Functors preserving finite products and satisfying a similar descent condition with respect to *hypercovers* (see [HM24, Def. A.4.19] for a precise definition) are called *hypersheaves*. We denote by $\mathsf{HSh}(\mathsf{C}, \mathsf{D})$ the full subcategory of $\mathsf{Sh}(\mathsf{C}, \mathsf{D})$ constituted of the hypersheaves.

Consider a sheaf of abelian groups $\mathscr{G}$ on C. Using the natural inclusion $\mathsf{Ab} \hookrightarrow \mathsf{D}(\mathsf{Ab})$, $\mathscr{G}$ can be viewed as a presheaf valued in the derived ∞-category $\mathsf{D}(\mathsf{Ab})$. While this often fails to satisfy descent, one can sheafify to obtain a functor $\mathsf{Ab}(\mathsf{C}) \to \mathsf{Sh}(\mathsf{C}, \mathsf{D}(\mathsf{Ab}))$. This induces a t-exact fully faithful functor $\mathsf{D}(\mathsf{Ab}(\mathsf{C})) \hookrightarrow \mathsf{Sh}(\mathsf{C}, \mathsf{D}(\mathsf{Ab}))$, whose essential image is precisely $\mathsf{HSh}(\mathsf{C}, \mathsf{D}(\mathsf{Ab}))$ [SAG, Cor. 2.1.2.3].

[4] Denoting the category of abelian sheaves on C by $\mathsf{Ab}(\mathsf{C})$ is an abuse of notation, since this is the category of abelian group objects *on the associated 1-topos*.

The distinction between hypersheaves and ordinary sheaves is a phenomenon unique to ∞-categories. More precisely, the inclusion $\mathsf{HSh}(\mathsf{C},\mathsf{D}) \hookrightarrow \mathsf{Sh}(\mathsf{C},\mathsf{D})$ is an equivalence as soon as D is a n-category for some *finite* n [HTT, Lem. 6.5.2.9]. Consequently,

$$\mathsf{D}^{[-1,0]}(\mathsf{Ab}(\mathsf{C})) \to \mathsf{Sh}(\mathsf{C},\mathsf{D}^{[-1,0]}(\mathsf{Ab}))$$

is an equivalence of ∞-categories.

Finally, recall that the Dold–Kan correspondence gives an equivalence of ∞-categories $\mathsf{D}^{\leqslant 0}(\mathsf{Ab}) \xrightarrow{\sim} \mathsf{An}(\mathsf{Ab})$. Here, the i-th homotopy group of an object on the right is isomorphic to the $-i$-th cohomology group of the corresponding object on the left. In particular, this restricts to an equivalence $\mathsf{D}^{[-1,0]}(\mathsf{Ab}) \xrightarrow{\sim} \mathsf{An}_{\leqslant 1}(\mathsf{Ab})$ leading to the desired equivalence $\mathsf{D}^{[-1,0]}(\mathsf{Ab}(\mathsf{C})) \xrightarrow{\sim} \mathsf{Sh}(\mathsf{C},\mathsf{An}_{\leqslant 1}(\mathsf{Ab}))$. □

Let us momentarily denote by $\mathcal{G}^\circ$ the object in the derived category of abelian sheaves associated with a commutative group stack $\mathcal{G}$. (We will soon begin identifying these objects.) Here, we outline some consequences of the proof of Proposition 2.3.

■ Remark 2.4.

1. Let $\mathscr{G}$ be an abelian sheaf. Composing with the natural inclusion $\mathsf{Ab} \hookrightarrow \mathsf{An}_{\leqslant 1}(\mathsf{Ab})$, we obtain a commutative group stack denoted by the same symbol. Then, $\mathscr{G}^\circ$ is isomorphic to the complex $[0 \to \mathscr{G}]$. Similarly, $(\mathsf{B}\mathscr{G})^\circ$ is isomorphic to $[\mathscr{G} \to 0]$.

2. Given a commutative group stack $\mathcal{G}$, the abelian sheaf $\mathscr{H}^0(\mathcal{G}^\circ)$ is isomorphic to the coarse moduli sheaf of $\mathcal{G}$. This is the sheafification of the presheaf sending an object X of C to the group of isomorphism classes $\pi_0(\mathcal{G}(X))$ of $\mathcal{G}(X)$. Similarly, $\mathscr{H}^{-1}(\mathcal{G}^\circ)$ is isomorphic to the automorphism sheaf of a zero section of $\mathcal{G}$.

3. Let $\mathcal{G}$ and $\mathcal{H}$ be commutative group stacks. We denote by $\underline{\mathrm{Hom}}(\mathcal{G},\mathcal{H})$ the internal Hom of commutative group stacks. This is another commutative group stack satisfying $\underline{\mathrm{Hom}}(\mathcal{G},\mathcal{H})^\circ \simeq \tau_{\leqslant 0}\mathsf{R}\underline{\mathrm{Hom}}(\mathcal{G}^\circ,\mathcal{H}^\circ)$. ■

Henceforth, we will focus on the big fppf site $\mathsf{C} = (\mathsf{Sch}/S)_{\mathrm{fppf}}$ associated with a base scheme S. There is a distinguished abelian sheaf on this site: the multiplicative group $\mathbb{G}_m$, sending a S-scheme T to the group of units $\Gamma(T,\mathcal{O}_T)^\times$. Its classifying stack, $\mathsf{B}\mathbb{G}_m$, will play a role in this story similar to that of the circle group in the Pontryagin duality of locally compact abelian groups.

Definition 2.5. Let $\mathcal{G}$ be a commutative group stack on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. We define its *Cartier dual* $\mathcal{G}^D$ as $\underline{\mathrm{Hom}}(\mathcal{G},\mathbb{G}_m)$ and its *stacky Cartier dual* $\mathcal{G}^\vee$ as $\underline{\mathrm{Hom}}(\mathcal{G},\mathsf{B}\mathbb{G}_m)$.

From our perspective, the stacky Cartier dual $\mathcal{G}^\vee$ is the *true* dual of a commutative group stack while $\mathcal{G}^D$ is a good enough approximation in some situations. The following remark explains our motivation for considering $\mathcal{G}^\vee$.

■ REMARK 2.6. Let $\mathcal{G}$ be a commutative group stack on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$, and denote is group law by $m\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$. Given an S-scheme T, the groupoid $\mathcal{G}^\vee(T)$ can be described as follows: its objects are pairs $(\mathscr{L}, \alpha)$, where $\mathscr{L}$ is a line bundle on $\mathcal{G} \times_S T$ and α is an isomorphism $m^*\mathscr{L} \to \mathscr{L} \boxtimes \mathscr{L}$ making diagrams (A) and (B) just above [Bro21, Rem. 3.13] commute. A morphism from $(\mathscr{L}, \alpha)$ to $(\mathscr{L}', \alpha')$ is an isomorphism of line bundles $\varphi\colon \mathscr{L} \to \mathscr{L}'$ making the diagram

$$\begin{array}{ccc} m^*\mathscr{L} & \xrightarrow{\alpha} & \mathscr{L} \boxtimes \mathscr{L} \\ {\scriptstyle m^*\varphi}\downarrow & & \downarrow{\scriptstyle \varphi\boxtimes\varphi} \\ m^*\mathscr{L}' & \xrightarrow{\alpha'} & \mathscr{L}' \boxtimes \mathscr{L}' \end{array}$$

commute. The symmetric monoidal structure on $\mathcal{G}^\vee(T)$ sends $(\mathscr{L}, \alpha)$ and $(\mathscr{L}', \alpha')$ to the pair $(\mathscr{L} \otimes \mathscr{L}', \alpha \cdot \alpha')$, where $\alpha \cdot \alpha'$ is defined as the composition

$$m^*(\mathscr{L} \otimes \mathscr{L}') \simeq m^*\mathscr{L} \otimes m^*\mathscr{L}' \xrightarrow{\alpha\otimes\alpha'} (\mathscr{L} \boxtimes \mathscr{L}) \otimes (\mathscr{L}' \boxtimes \mathscr{L}') \simeq (\mathscr{L} \otimes \mathscr{L}') \boxtimes (\mathscr{L} \otimes \mathscr{L}').$$

The reader might also be interested in comparing this description to [SGA 7_{I}, Exp. VII, §1.1.6] or [Mor85, §I.2.3]. ■

In most cases of interest in this paper, we will work with commutative group stacks $\mathcal{G}$ defined by complexes $[\mathscr{F} \to \mathscr{G}]$, where the map $d\colon \mathscr{F} \to \mathscr{G}$ is a monomorphism. In this case, $\mathcal{G} \simeq \operatorname{coker} d$ takes values in the full subcategory Ab of $\mathsf{An}_{\leqslant 1}(\mathsf{Ab})$. The following proposition gives a convenient criterion for computing the stacky Cartier duals of these objects.

PROPOSITION 2.7. *Let $\mathscr{G}$ be an abelian sheaf on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. Then the stacky Cartier dual of the classifying stack $\mathsf{B}\mathscr{G}$ is isomorphic to $\mathscr{G}^D$. If $\underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m) = 0$, then $\mathscr{G}^\vee \simeq \mathsf{B}\mathscr{G}^D$. Similarly, if $\mathscr{G}^D = 0$, then $\mathscr{G}^\vee \simeq \underline{\mathrm{Ext}}^1(\mathscr{G}, \mathbb{G}_m)$.*

Proof. The first isomorphism is simply the fact that $((\mathsf{B}\mathscr{G})^\vee)^\circ$ is given by

$$\tau_{\leqslant 0} R\underline{\mathrm{Hom}}(\mathscr{G}[1], \mathbb{G}_m[1]) \simeq \underline{\mathrm{Hom}}(\mathscr{G}, \mathbb{G}_m).$$

Similarly, using the explicit description of $(\mathscr{G}^\vee)^\circ$ as $\tau_{\leqslant 0} R\underline{\mathrm{Hom}}(\mathscr{G}, \mathbb{G}_m[1])$, the other isomorphisms follow from the fact that an object of a derived category, whose cohomology is concentrated in a single degree, is isomorphic to that cohomology in the corresponding degree. □

2.2. COMPUTATIONS OF CARTIER DUALS

Consider a base scheme S, and let $\mathscr{G}$ be an abelian sheaf on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$, regarded as a commutative group stack. Our approach to computing the stacky Cartier dual $\mathscr{G}^\vee$ predominantly follows the method outlined in Proposition 2.7. We typically either prove the vanishing of $\underline{\mathrm{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$ and calculate $\mathcal{G}^D$, or we establish that $\mathcal{G}^D$ vanishes and compute $\underline{\mathrm{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$. This section focuses on computing the Cartier duals $\mathcal{G}^D$ for some abelian sheaves $\mathcal{G}$ of interest.

PROPOSITION 2.8. *Let A be an abelian scheme and T be a torus over a base scheme S. Then the Cartier dual A^D of A vanishes, and the Cartier dual X of T is representable by a group scheme that is étale locally isomorphic to a finite product of copies of the constant group scheme $\mathbb{Z}$. Moreover, the Cartier dual of X is isomorphic to T.*

Proof. Denote by $p\colon A \to S$ the structure map, and let X be an S-scheme. By the universal property of the global spectrum, a morphism of schemes $A_X \to \mathbb{G}_{m,X}$ over X is equivalent to a morphism of $\mathcal{O}_X$-algebras

$$\mathcal{O}_X[x, x^{-1}] \to p_{X,*}\mathcal{O}_{A_X} \simeq \mathcal{O}_X,$$

where $p_X\colon A_X \to X$ is the base change of p [Stacks, Tag 0E0L]. It follows that every morphism of schemes $A_X \to \mathbb{G}_{m,X}$ over X must be constant. If it is a morphism of groups, it has to be trivial. This proves that A^D vanishes. The statements about tori are proven in [SGA 3_{II}, Exp. X, Cor. 5.7]. □

Moving forward, we focus on the case where the base scheme S is the spectrum of a characteristic zero field k. The following proposition recalls the standard setting for Cartier duality [SGA 3_{I}, Exp. $\mathrm{VII_B}$, Prop. 2.2.2]. (Proposition 2.9 is independent of the characteristic of k, but every other result below requires it to be zero.)

PROPOSITION 2.9. *Let $G = \operatorname{Spec} R$ be an affine commutative group scheme over k. Its Cartier dual G^D is represented by the formal group $\operatorname{Spf} R^*$, where R^* is the dual Hopf algebra. Moreover, the double dual $(G^D)^D$ is naturally isomorphic to G.*

Let U be a commutative unipotent algebraic group over k. Since the exponential map gives an isomorphism between U and its Lie algebra, U is necessarily a vector group [Mil17, Prop. 14.32] and we denote by U^* its vector space dual.

PROPOSITION 2.10. *Let U be a commutative unipotent algebraic group with dual U^*. Denote by $\widehat{U}$ and $\widehat{U^*}$ their formal completions along the zero sections. Then $U^D \simeq \widehat{U^*}$ and $\widehat{U}^D \simeq U^*$.*

Proof. The computation $U^D \simeq \widehat{U^*}$ follows from the fact that the dual of $\mathrm{Sym}(U^*)$ is the completion of $\mathrm{Sym}(U)$ at the ideal of degree one elements. (Upon a choice of basis, this is nothing but the isomorphism $k[x_1, \ldots, x_n]^* \simeq k[\![x_1, \ldots, x_n]\!]$.) The other computation follows from this one by duality. □

Given a k-algebra R, we have that $\widehat{\mathbb{G}}_a(R)$ is the group of nilpotent elements in R and $\widehat{\mathbb{G}}_m(R)$ is that of unipotent elements. We remark that the formal groups $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$ are isomorphic via the map

$$\begin{aligned}\widehat{\mathbb{G}}_m(R) &\to \widehat{\mathbb{G}}_a(R)\\ 1+x &\mapsto \log(1+x).\end{aligned}$$

This phenomenon is a general property of formal groups in characteristic zero, and it simplifies their study.

PROPOSITION 2.11 (Cartier). *Let* G *be a commutative algebraic group over* k *and let* $\mathfrak{g}$ *be its Lie algebra, seen as a vector group. The formal completions of* G *and of* $\mathfrak{g}$ *along their zero sections coincide.*

Proof. Since an algebraic group and its formal completion share the same Lie algebra, the composition

$$\left\{\begin{array}{c}\text{Algebraic}\\ \text{groups over } k\end{array}\right\} \xrightarrow{\widehat{(-)}} \left\{\begin{array}{c}\text{Infinitesimal formal}\\ \text{groups over } k\end{array}\right\} \xrightarrow{\mathrm{Lie}(-)} \left\{\begin{array}{c}\text{Lie algebras}\\ \text{over } k\end{array}\right\}$$

is the functor associating an algebraic group to its Lie algebra. In particular, G and $\mathfrak{g}$, seen as a vector group, have the same image by the composition above. Now, by [SGA 3_I, Exp. VII_B, Cor. 3.3.2], the functor on the right is an equivalence of categories. In particular, G and $\mathfrak{g}$ have isomorphic formal completions. □

The preceding propositions enable us to compute the Cartier dual of the formal completions of commutative algebraic groups. This result, with a slightly different proof, also appears in [BB09, Lem. A.3.1].

COROLLARY 2.12. *Let* G *be a commutative algebraic group over* k *and denote by* $\mathfrak{g}$ *its Lie algebra. The Cartier dual of the formal completion* $\widehat{G}$ *is naturally isomorphic to* $\mathfrak{g}^*$. *This also coincides with the* invariant *differentials* Ω_G *of* G.

We are now in position to compute the Cartier dual of the de Rham space $G_{dR} \simeq G/\widehat{G}$ of a commutative connected algebraic group G.

PROPOSITION 2.13. *For a commutative connected algebraic group* G *over* k, *the Cartier dual of* G_{dR} *vanishes.*

Proof. Recall from the Barsotti–Chevalley theorem [Mil17, Thm. 8.28 and Cor. 16.15] that G is an extension of an abelian variety A by a product of a torus T and a unipotent group U. Since the de Rham functor $(-)_{dR}$ is exact, the de Rham space G_{dR} is an extension of A_{dR} by $T_{dR} \times U_{dR}$. Thus, we have an induced exact sequence

$$0 \to \underline{\mathrm{Hom}}(A_{dR}, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(G_{dR}, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(T_{dR}, \mathbb{G}_m) \times \underline{\mathrm{Hom}}(U_{dR}, \mathbb{G}_m),$$

and so it suffices to prove the result when G is an abelian variety, a torus, or a unipotent group.

Because the functor $\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ is left-exact, the vanishing of G_{dR}^D is equivalent to the morphism $\underline{\mathrm{Hom}}(G, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(\widehat{G}, \mathbb{G}_m)$ being a monomorphism. We verify this in the relevant particular cases. For abelian varieties, their vanishing Cartier dual ensures this property. For a unipotent group U, the relevant morphism is isomorphic to $\widehat{U^*} \to U^*$, which is likewise monic per Corollary A.5.

For a torus T, the question of whether $\underline{\mathrm{Hom}}(T, \mathbb{G}_m) \to \underline{\mathrm{Hom}}(\widehat{T}, \mathbb{G}_m)$ is a monomorphism is local on the base, allowing us to assume that $T = \mathbb{G}_m$. Then the statement boils down to the following: given a connected k-algebra R, if

$$\begin{array}{ccccc} \mathrm{Uni}(B) & \hookrightarrow & B^\times & \to & B^\times \\ x & \mapsto & x & \mapsto & x^n \end{array}$$

is the unit map for every R-algebra B, then $n = 0$. Here, $\mathrm{Uni}(B)$ denotes the group of unipotent elements in B. This can be proven by choosing $B = R[z]/(z-1)^r$ for some $r > n$, leading to the desired result. □

The hypothesis that G is connected in the Proposition 2.13 is essential. In general, a commutative algebraic group G over k fits into a short exact sequence

$$0 \to G^0 \to G \to \pi_0(G) \to 0,$$

where G^0 is the connected component of G containing the identity, and $\pi_0(G)$ is the finite étale group of connected components. Proposition A.6 states that the natural map $\pi_0(G) \to \pi_0(G)_{dR}$ is an isomorphism, and the preceding result then implies that the Cartier duals of G_{dR} and $\pi_0(G)$ coincide.

2.3. Extensions by the multiplicative group

As in the previous section, consider a base scheme S and an abelian sheaf $\mathcal{G}$ on $(\mathsf{Sch}/S)_{fppf}$. To continue our strategy outlined in the beginning of Section 2.2, we now focus on computing the extension sheaves $\underline{\mathrm{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$. We start by recalling some fundamental computations.

Proposition 2.14. *Let A be an abelian scheme and T be a torus over S with Cartier dual X. Then $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ is isomorphic to the dual abelian scheme of A, while both $\underline{\mathrm{Ext}}^1(T, \mathbb{G}_m)$ and $\underline{\mathrm{Ext}}^1(X, \mathbb{G}_m)$ vanish.*

Proof. The identification of $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ with the dual abelian scheme is commonly referred to as the *Barsotti–Weil formula*. A complete proof may be found in [RR25, Cor. 3.6]. (Alternatively, this identification also follows directly from Lemma 2.21.) The vanishing of $\underline{\mathrm{Ext}}^1(T, \mathbb{G}_m)$ was established in [SGA 7_{I}, Exp. VIII, Prop. 3.3.1]. Finally, the vanishing of $\underline{\mathrm{Ext}}^1(X, \mathbb{G}_m)$ is local on S, so we may reduce to the case $X \simeq \mathbb{Z}$. In this case, the claim follows from the exactness of the functor $\underline{\mathrm{Hom}}(\mathbb{Z}, -)$, which is isomorphic to the identity functor. □

We now describe the strategy of [RR25, §2] for computing extension sheaves, specialized to our setting. Let $\mathscr{G}$ and $\mathscr{A}$ be abelian sheaves on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. As $\underline{\mathrm{Ext}}^i(\mathscr{G}, \mathscr{A})$ is the sheafification of the presheaf $X \mapsto \mathrm{Ext}^i_X(\mathscr{G}, \mathscr{A})$ [SGA 4_{I}, Prop. V.6.1], there is a natural morphism of groups

$$\mathrm{Ext}^i_X(\mathscr{G}, \mathscr{A}) \to \underline{\mathrm{Ext}}^i(\mathscr{G}, \mathscr{A})(X),$$

functorial on $\mathscr{G}$, $\mathscr{A}$, and X. The following simple result, also contained in [SGA 4_{I}, Prop. V.6.1], gives a relation between these objects

Proposition 2.15. *Let X be a scheme over S and $\mathscr{G}, \mathscr{A}$ be abelian sheaves on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. There exists an exact sequence*

$$0 \to H^1(X, \underline{\mathrm{Hom}}(\mathscr{G}, \mathscr{A})) \to \mathrm{Ext}^1_X(\mathscr{G}, \mathscr{A}) \to \underline{\mathrm{Ext}}^1(\mathscr{G}, \mathscr{A})(X) \to H^2(X, \underline{\mathrm{Hom}}(\mathscr{G}, \mathscr{A})),$$

that is functorial on $\mathscr{G}$, $\mathscr{A}$, and X.

Proof. The group of sections of $\underline{\mathrm{Hom}}(\mathscr{G}, \mathscr{A})$ over X is, by definition, $\mathrm{Hom}_X(\mathscr{G}, \mathscr{A})$. In particular, there is a Grothendieck spectral sequence, often called *local-to-global spectral sequence*, whose five-term exact sequence yields the desired result. □

Now that we have established a connection between extension *sheaves* and extension *groups*, we will study a method for computing the latter. The following proposition, initially suggested by Grothendieck in [SGA 7_{I}, Exp. VII, Rem. 3.5.4] and partially developed by Breen in [Bre69], has been independently proven by Deligne (in a letter to Breen available in [Rib24, App. B]) and by Clausen–Scholze [SC19, Thm. 4.10].

Proposition 2.16 (Breen–Deligne resolution). *Let $\mathscr{G}$ be an abelian sheaf on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. There exists a functorial resolution of the form*

$$\cdots \to \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathscr{G}^{r_{i,j}}] \to \cdots \to \mathbb{Z}[\mathscr{G}^3] \oplus \mathbb{Z}[\mathscr{G}^2] \to \mathbb{Z}[\mathscr{G}^2] \to \mathbb{Z}[\mathscr{G}] \to \mathscr{G},$$

where the n_i and $r_{i,j}$ are all positive integers.

Clausen and Scholze's proof shows that the first terms of the resolution can be chosen in the following way:

$$\mathbb{Z}[\mathscr{G}^4] \oplus \mathbb{Z}[\mathscr{G}^3] \oplus \mathbb{Z}[\mathscr{G}^3] \oplus \mathbb{Z}[\mathscr{G}^2] \oplus \mathbb{Z}[\mathscr{G}] \xrightarrow{d_3} \mathbb{Z}[\mathscr{G}^3] \oplus \mathbb{Z}[\mathscr{G}^2] \xrightarrow{d_2} \mathbb{Z}[\mathscr{G}^2] \xrightarrow{d_1} \mathbb{Z}[\mathscr{G}] \xrightarrow{d_0} \mathscr{G},$$

where the differentials are given by

$$\begin{aligned}
d_3([x,y,z,t]) &= ([x+y,z,t]-[x,y+z,t]+[x,y,z+t]-[x,y,z]-[y,z,t],0)\\
d_3([x,y,z]) &= (-[x,y,z]+[x,z,y]-[z,x,y],[x+y,z]-[x,z]-[y,z])\\
d_3([x,y,z]) &= ([x,y,z]-[y,x,z]+[y,z,x],[x,y+z]-[x,y]-[x,z])\\
d_3([x,y]) &= (0,[x,y]+[y,x])\\
d_3([x]) &= (0,[x,x])\\
d_2([x,y,z]) &= [x+y,z]-[x,y+z]+[x,y]-[y,z]\\
d_2([x,y]) &= [x,y]-[y,x]\\
d_1([x,y]) &= [x+y]-[x]-[y]\\
d_0([x]) &= x.
\end{aligned}$$

Here, the top $d_3([x,y,z])$ acts on the first factor of $\mathbb{Z}[\mathscr{G}^3]$, while the bottom $d_3([x,y,z])$ acts on the second factor. Henceforth, we fix a Breen–Deligne resolution that begins with these terms. In particular, this explicit description allows us to define two important invariants.

Definition 2.17. Let $\mathscr{G}$ and $\mathscr{A}$ be abelian sheaves on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. Applying the functor $\mathrm{Hom}(-,\mathscr{A})$ to the Breen–Deligne resolution of $\mathscr{G}$, we obtain the complex

$$\Gamma(\mathscr{G},\mathscr{A}) \to \Gamma(\mathscr{G}^2,\mathscr{A}) \to \Gamma(\mathscr{G}^3,\mathscr{A}) \oplus \Gamma(\mathscr{G}^2,\mathscr{A}) \to \Gamma(\mathscr{G}^4,\mathscr{A}) \oplus \Gamma(\mathscr{G}^3,\mathscr{A}) \oplus \Gamma(\mathscr{G}^3,\mathscr{A}) \oplus \Gamma(\mathscr{G}^2,\mathscr{A}) \oplus \Gamma(\mathscr{G},\mathscr{A}).$$

We denote the first cohomology of this complex by $H^2_s(\mathscr{G},\mathscr{A})$ and the second cohomology by $H^3_s(\mathscr{G},\mathscr{A})$.

Let $\mathscr{X}$ be a sheaf of sets and let $\mathscr{A}$ be an abelian sheaf on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. We define the cohomology group $H^i(\mathscr{X},\mathscr{A})$ as the value at $\mathscr{A}$ of the i-th right derived functor of

$$\mathrm{Mor}(\mathscr{X},-)\colon \mathsf{Ab}((\mathsf{Sch}/S)_{\mathrm{fppf}}) \to \mathsf{Ab}.$$

If $\mathscr{X}$ is representable by an S-scheme X, the Yoneda lemma implies that $H^i(\mathscr{X},\mathscr{A})$ coincides with the usual cohomology group $H^i(X,\mathscr{A})$.

As usual, the group $H^1(\mathscr{X},\mathscr{A})$ classifies $\mathscr{A}$-torsors over $\mathscr{X}$, where the group law is given by the contracted product. Moreover, any morphism $f\colon \mathscr{X}\to\mathscr{Y}$ induces a homomorphism

$$f^*\colon H^1(\mathscr{Y},\mathscr{A}) \to H^1(\mathscr{X},\mathscr{A}),$$

which sends an $\mathscr{A}$-torsor $\mathscr{P} \to \mathscr{Y}$ to its pullback $\mathscr{P} \times_{\mathscr{Y}} \mathscr{X} \to \mathscr{X}$. When $\mathscr{X} = \mathscr{G}$ is itself an abelian sheaf, we define $H^1_m(\mathscr{G}, \mathscr{A})$ to be the subgroup of $H^1(\mathscr{G}, \mathscr{A})$ consisting of those $\mathscr{A}$-torsors over $\mathscr{G}$ that are compatible with the group structure on $\mathscr{G}$.

Definition 2.18. Let $\mathscr{G}$ and $\mathscr{A}$ be abelian sheaves on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. Denote by $m\colon \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ the group operation of $\mathscr{G}$, and by $\mathrm{pr}_1, \mathrm{pr}_2\colon \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ the natural projections. We define $H^1_m(\mathscr{G}, \mathscr{A})$ as the kernel of the morphism $m^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*$.

Put simply, $H^1_m(\mathscr{G}, \mathscr{A})$ is the group of isomorphism classes of $\mathscr{A}$-torsors $\mathscr{P}$ over $\mathscr{G}$ satisfying $m^*\mathscr{P} \simeq \mathrm{pr}_1^*\mathscr{P} \wedge \mathrm{pr}_2^*\mathscr{P}$. Often, we say that these $\mathscr{A}$-torsors are *multiplicative*. With this terminology established, we may now explain the computation of the extension groups.

Proposition 2.19 (Breen). *Let $\mathscr{G}$ and $\mathscr{A}$ be abelian sheaves on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. There exists an exact sequence*

$$0 \to H^2_s(\mathscr{G}, \mathscr{A}) \to \mathrm{Ext}^1(\mathscr{G}, \mathscr{A}) \to H^1_m(\mathscr{G}, \mathscr{A}) \to H^3_s(\mathscr{G}, \mathscr{A}) \to \mathrm{Ext}^2(\mathscr{G}, \mathscr{A}),$$

that is functorial in $\mathscr{G}$ and $\mathscr{A}$.

Before diving into the proof, let us explain the morphism $\mathrm{Ext}^1(\mathscr{G}, \mathscr{A}) \to H^1_m(\mathscr{G}, \mathscr{A})$. Since $\mathrm{Ext}^1(-, \mathscr{A})$ is an additive functor, an extension $\mathscr{E}$ of $\mathscr{G}$ by $\mathscr{A}$ always satisfies

$$m^*\mathscr{E} = (\mathrm{pr}_1 + \mathrm{pr}_2)^*\mathscr{E} \simeq \mathrm{pr}_1^*\mathscr{E} + \mathrm{pr}_2^*\mathscr{E},$$

where the sum on the right is the Baer sum of extensions. Such an extension defines an $\mathscr{A}$-torsor[5] $\mathscr{P}$ over $\mathscr{G}$, which satisfies $m^*\mathscr{P} \simeq \mathrm{pr}_1^*\mathscr{P} \wedge \mathrm{pr}_2^*\mathscr{P}$.

Proof of Proposition 2.19. The universal property of free objects gives that $H^i(\mathscr{G}^n, \mathscr{A})$ is isomorphic to $\mathrm{Ext}^i(\mathbb{Z}[\mathscr{G}^n], \mathscr{A})$ for all n and i. Then, the Breen–Deligne resolution yields a spectral sequence

$$E_1^{i,j}\colon \prod_{r=1}^{n_i} H^j(\mathscr{G}^{s_{i,r}}, \mathscr{A}) \implies \mathrm{Ext}^{i+j}(\mathscr{G}, \mathscr{A}),$$

whose five-term exact sequence is precisely the one in the statement. □

■ Remark 2.20. In [SGA 7_{I}, Exp. VII, §1.2], Grothendieck proved that $\mathrm{Ext}^1(\mathscr{G}, \mathscr{A})$ is isomorphic to the group of isomorphism classes of pairs $(\mathscr{P}, \alpha)$, where $\mathscr{P}$ is a $\mathscr{A}$-torsor over $\mathscr{G}$ and $\alpha\colon m^*\mathscr{P} \to \mathrm{pr}_1^*\mathscr{P} \wedge \mathrm{pr}_2^*\mathscr{P}$ is an isomorphism of $\mathscr{A}$-torsors over $\mathscr{G} \times \mathscr{G}$ making

[5] We refer the reader to Remark 3.4 for more explanations.

two diagrams (imposing that $\mathscr{P}$ admits an associative and commutative group law) commute. In particular, our invariants $H^2_s(\mathscr{G}, \mathscr{A})$ and $H^3_s(\mathscr{G}, \mathscr{A})$ govern how far the map

$$\begin{aligned} \mathrm{Ext}^1(\mathscr{G}, \mathscr{A}) &\to H^1_m(\mathscr{G}, \mathscr{A}) \\ [\mathscr{P}, \alpha] &\mapsto [\mathscr{P}] \end{aligned}$$

is from being an isomorphism. ∎

Even though the first terms of the Breen–Deligne resolution are explicit, the invariants $H^2_s(\mathscr{G}, \mathscr{A})$ and $H^3_s(\mathscr{G}, \mathscr{A})$ are usually quite hard to compute. The following observation will suffice for their computations in many interesting cases.

Lemma 2.21. *Let $\mathscr{G}$ and $\mathscr{A}$ be abelian sheaves on $(\mathsf{Sch}/S)_{\mathrm{fppf}}$. If every morphism $\mathscr{G}^n \to \mathscr{A}$ of sheaves of* sets *(for $n = 2, 3$) can be expressed as a sum of maps $\mathscr{G} \to \mathscr{A}$, then both $H^2_s(\mathscr{G}, \mathscr{A})$ and $H^3_s(\mathscr{G}, \mathscr{A})$ vanish.*

Proof. First, we demonstrate that $H^2_s(\mathscr{G}, \mathscr{A})$ vanishes. The kernel of

$$\Gamma(\mathscr{G}^2, \mathscr{A}) \to \Gamma(\mathscr{G}^3, \mathscr{A}) \oplus \Gamma(\mathscr{G}^2, \mathscr{A})$$

is composed of the maps $f\colon \mathscr{G}^2 \to \mathscr{A}$ that satisfy $f(x+y, z) - f(y, z) = f(x, y+z) - f(x, y)$ and $f(x, y) = f(y, x)$. By applying the given hypothesis, we find morphisms $f_1, f_2\colon \mathscr{G} \to \mathscr{A}$ such that $f(x, y) = f_1(x) + f_2(y)$. This simplifies the first equation to

$$f_1(x+y) - f_1(y) = f_2(y+z) - f_2(y).$$

By setting $x = y = 0$ and $y = z = 0$, we observe that f must be constant. Next, the image of

$$\Gamma(\mathscr{G}, \mathscr{A}) \to \Gamma(\mathscr{G}^2, \mathscr{A})$$

consists of maps of the form $(x, y) \mapsto g(x+y) - g(x) - g(y)$, for some $g\colon \mathscr{G} \to \mathscr{A}$. Since every constant map is of this form, it follows that the cohomology vanishes.

We will use the same strategy to show that $H^3_s(\mathscr{G}, \mathscr{A})$ also vanishes. The kernel of the morphism

$$\Gamma(\mathscr{G}^3, \mathscr{A}) \oplus \Gamma(\mathscr{G}^2, \mathscr{A}) \to \Gamma(\mathscr{G}^4, \mathscr{A}) \oplus \Gamma(\mathscr{G}^3, \mathscr{A}) \oplus \Gamma(\mathscr{G}^3, \mathscr{A}) \oplus \Gamma(\mathscr{G}^2, \mathscr{A}) \oplus \Gamma(\mathscr{G}, \mathscr{A})$$

is composed of the maps $p\colon \mathscr{G}^3 \to \mathscr{A}$ and $q\colon \mathscr{G}^2 \to \mathscr{A}$ satisfying

$$p(x, y, z) + p(x, y+z, t) + p(y, z, t) = p(x+y, z, t) + p(x, y, z+t) \tag{1.1}$$

$$p(x, y, z) + p(z, x, y) + q(x, z) + q(y, z) = p(x, z, y) + q(x+y, z) \tag{2.1}$$

$$p(x, y, z) + p(y, z, x) + q(x, y+z) = p(y, x, z) + q(x, y) + q(x, z) \tag{3.1}$$

$$q(x, y) + q(y, x) = 0 \tag{4.1}$$

$$q(x, x) = 0. \tag{5.1}$$

To organize the remainder of the proof, we will denote a simplified version of the Equation (n.i) as (n.i + 1). Once again, we write $p(x, y, z) = p_1(x) + p_2(y) + p_3(z)$ for some $p_i\colon \mathscr{G} \to \mathscr{A}$, and $q(x, y) = q_1(x) + q_2(y)$ for some $q_i\colon \mathscr{G} \to \mathscr{A}$. Using Equation (5.1), we replace every instance of q_2 by $-q_1$. This yields the following relations.

$$p_3(z) + p_1(x) + p_2(y + z) + p_1(y) + p_3(t) = p_1(x + y) + p_3(z + t) \tag{1.2}$$

$$p_2(y) + p_3(z) + p_1(z) + p_2(x) + q_1(x) + q_1(y) = p_2(z) + q_1(x + y) + q_1(z) \tag{2.2}$$

$$p_1(x) + p_2(y) + p_2(z) + p_3(x) + q_1(y) + q_1(z) = p_2(x) + q_1(y + z) + q_1(x) \tag{3.2}$$

By setting $x = y = 0$ in Equations (2.2) and (3.2), we obtain

$$p_2(0) + p_3(z) + p_1(z) + q_1(0) = q_1(z) \tag{2.3}$$

$$p_1(0) + p_2(z) + p_3(0) = 0. \tag{3.3}$$

With this information, we can simplify Equation (1.2):

$$p_3(z) + p_1(x) + p_2(0) + p_1(y) + p_3(t) = p_1(x + y) + p_3(z + t). \tag{1.3}$$

Restricting this equation to $x = 0$ and $t = 0$ gives two new relations, implying that p_1 and p_3 are essentially morphisms of groups up to constants.

$$p_1(x) + p_1(y) - p_1(0) = p_1(x + y) \tag{1.4}$$

$$p_3(z) + p_3(t) - p_3(0) = p_3(z + t) \tag{1.5}$$

In summary, we have obtained the following equations:

$$p_1(x) + p_1(y) - p_1(0) = p_1(x + y) \tag{1.4}$$

$$p_3(z) + p_3(t) - p_3(0) = p_3(z + t) \tag{1.5}$$

$$p_1(z) - p_1(0) + p_3(z) - p_3(0) = q_1(z) - q_1(0) \tag{2.3}$$

$$p_1(0) + p_2(z) + p_3(0) = 0. \tag{3.3}$$

Finally, to prove that $H^3_s(\mathscr{G}, \mathscr{A})$ vanishes, we need to find a morphism $h\colon \mathscr{G}^2 \to \mathscr{A}$ such that

$$\begin{aligned} p(x, y, z) &= h(x + y, z) - h(x, y + z) + h(x, y) - h(y, z) \\ q(x, y) &= h(x, y) - h(y, x). \end{aligned}$$

A quick verification using the previously established equations shows that the morphism $h(x, y) := p_1(x) - p_3(y)$ satisfies this criterion. □

The following proposition extends a result by Colliot-Thélène and Gabber [Col08, Prop. 3.2 and Thm. 5.6]. Their method of proof can also be applied to our setting, but this result follows directly from our machinery.

Theorem 2.22. *Let* G *be a commutative group scheme over a reduced base* S*. Assume that the morphism* $G \to S$ *is smooth and has connected geometric fibers. Then the natural map*

$$\mathrm{Ext}^1_S(G, \mathbb{G}_m) \to H^1_m(G, \mathbb{G}_m)$$

is an isomorphism.

Proof. This result is a direct application of Lemma 2.21, using González-Avilés' generalization of Rosenlicht's lemma [Gon17, Thm. 1.1]. For the reader's convenience, we note that this result holds more generally in the following context: G is a commutative group scheme over a reduced base S such that $G \to S$ is syntomic and has reduced, connected maximal geometric fibers. Here, a maximal geometric fiber of $G \to S$ is a scheme $G \times_S \mathrm{Spec}\,\overline{\kappa(\eta)}$, for some generic point η of an irreducible component of S. □

■ Remark 2.23. The hypothesis that the base S be reduced is essential in the theorem above. Indeed, let R be a reduced ring and set $R' := R[\varepsilon]/(\varepsilon^2)$. As explained in [RR25, Rem. 7.3], there is a natural isomorphism

$$H^2_s(\mathbb{G}_{a,R'}, \mathbb{G}_{m,R'}) \simeq H^2_s(\mathbb{G}_{a,R}, \mathbb{G}_{a,R}).$$

The right-hand side was explicitly computed in [RR25, Prop. 7.1] and is shown to be a free R-module whose rank equals the number of prime numbers that are not invertible in R. It follows that, unless R is a $\mathbb{Q}$-algebra, the natural map

$$\mathrm{Ext}^1_{R'}(\mathbb{G}_a, \mathbb{G}_m) \to H^1_m(\mathbb{G}_a, \mathbb{G}_m)$$

is not injective. ■

Henceforth, we concentrate on the case where the base scheme is the spectrum of a characteristic zero field. We start by using Proposition 2.15 to prove that the sheafification map is an isomorphism in certain cases.

Proposition 2.24. *Let* T *be a torus and* U *be a unipotent group over a characteristic zero field* k*. The sheafification map* $\mathrm{Ext}^1_S(T, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(T, \mathbb{G}_m)(S)$ *is an isomorphism for an irreducible geometrically unibranch* k*-scheme* S*. Similarly, the map* $\mathrm{Ext}^1_S(U, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)(S)$ *is an isomorphism for a* k*-scheme* S *that is either reduced or affine.*

Proof. Let S be an irreducible geometrically unibranch k-scheme. Proposition 2.14 states that the sheaf $\underline{\mathrm{Ext}}^1(T, \mathbb{G}_m)$ vanishes, so the desired result is equivalent to the vanishing of the cohomology group $H^1(S, X) = 0$, where $X := T^D$ is the Cartier dual of T. This holds due to [SGA 7_I, Exp. VIII, Prop. 5.1].

For the unipotent case, we may assume that $U = \mathbb{G}_a$. If S is an affine scheme, [Bha22, Rem. 2.2.18] implies that $H^i(S, \widehat{\mathbb{G}}_a)$ vanishes for all $i \neq 0$. We claim that, for a reduced k-scheme S, the cohomology groups $H^i(S, \widehat{\mathbb{G}}_a)$ vanish for all i. First, we prove that the *étale* cohomology groups $H^i_{ét}(S, \widehat{\mathbb{G}}_a)$ vanish, and then we identify these groups with their fppf analogues.

Recall that $\widehat{\mathbb{G}}_a(\operatorname{Spec} R)$ is the nilradical of a k-algebra R. Thus, the sheaf condition implies that $\widehat{\mathbb{G}}_a(S)$ vanishes for reduced S. Next, the cohomology $H^i_{ét}(S, \widehat{\mathbb{G}}_a)$ can be computed on the small étale site of S and, according to [Stacks, Tag 03PC.(8)], the restriction of $\widehat{\mathbb{G}}_a$ to this site vanishes.

Since $\mathbb{G}_{a,dR}$ is the étale quotient of $\mathbb{G}_a$ by $\widehat{\mathbb{G}}_a$, we obtain the following morphism of long exact sequences:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Gamma(S, \widehat{\mathbb{G}}_a) & \longrightarrow & \Gamma(S, \mathbb{G}_a) & \longrightarrow & \Gamma(S, \mathbb{G}_{a,dR}) & \longrightarrow & H^1(S, \widehat{\mathbb{G}}_a) & \longrightarrow & H^1(S, \mathbb{G}_a) & \longrightarrow \cdots \\ & & \| & & \| & & \| & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \Gamma(S, \widehat{\mathbb{G}}_a) & \longrightarrow & \Gamma(S, \mathbb{G}_a) & \longrightarrow & \Gamma(S, \mathbb{G}_{a,dR}) & \longrightarrow & H^1_{ét}(S, \widehat{\mathbb{G}}_a) & \longrightarrow & H^1_{ét}(S, \mathbb{G}_a) & \longrightarrow \cdots. \end{array}$$

Given that S is reduced, we have shown that $H^i_{ét}(S, \widehat{\mathbb{G}}_a) = 0$ for all i. Moreover, since $\mathbb{G}_a$ is smooth, the natural map $H^i(S, \mathbb{G}_a) \to H^i_{ét}(S, \mathbb{G}_a)$ is an isomorphism. Using these facts, a diagram chase gives that $H^i(S, \widehat{\mathbb{G}}_a)$ vanishes for all i. □

In order to analyze the sheafification map $\operatorname{Ext}^1_S(G, \mathbb{G}_m) \to \underline{\operatorname{Ext}}^1(G, \mathbb{G}_m)(S)$ for a connected commutative algebraic group G over a field of characteristic zero, we require the following vanishing result. We refer the reader to [RR25, §6] for further discussion of the elusive sheaf $\underline{\operatorname{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)$; in particular, we note that it does not vanish.

Proposition 2.25. *Let U be a unipotent group over a characteristic zero field k. Then the group $\underline{\operatorname{Ext}}^1(U, \mathbb{G}_m)(S)$ vanishes for all seminormal k-schemes S.*

Proof. Since seminormal schemes are reduced [Stacks, Tag 0EUQ], Theorem 2.22 and Proposition 2.24 imply that $\underline{\operatorname{Ext}}^1(U, \mathbb{G}_m)(S) \simeq H^1_m(U_S, \mathbb{G}_{m,S})$. Denoting by $p\colon U_S \to S$ the structure map, Traverso's theorem [Sad21, Lem. 4.3] gives that

$$p^*\colon H^1(S, \mathbb{G}_m) \to H^1(U_S, \mathbb{G}_m)$$

is an isomorphism. In particular, $H^1_m(U_S, \mathbb{G}_m)$ is isomorphic to the subgroup of $H^1(S, \mathbb{G}_m)$ consisting of elements $x \in H^1(S, \mathbb{G}_m)$ that satisfy $p^*x \in H^1_m(U_S, \mathbb{G}_m)$. Next, denote by $m\colon U_S \times_S U_S \to U_S$ the group operation and by $\mathrm{pr}_i\colon U_S \times_S U_S \to U_S$ the natural projections. Then, p^*x lies in $H^1_m(U_S, \mathbb{G}_m)$ if and only if

$$m^*p^*x = \mathrm{pr}_1^* p^*x + \mathrm{pr}_2^* p^*x.$$

However, the morphisms $p \circ m$, $p \circ \mathrm{pr}_1$, and $p \circ \mathrm{pr}_2$ are all equal to the structure map of U^2_S, which has a section $S \to U^2_S$. Thus, $m^*p^*x = \mathrm{pr}_1^* p^*x + \mathrm{pr}_2^* p^*x$ holds if and only if $x = 0$, completing the proof. □

We are now in position to prove an enhancement of Theorem 2.22 for commutative connected algebraic groups over a characteristic zero field.

Theorem 2.26. *Let* G *be a commutative connected algebraic group over a characteristic zero field* k. *For a regular* k*-scheme* S*, the natural maps*

$$\underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)(S) \leftarrow \mathrm{Ext}^1_S(G,\mathbb{G}_m) \to H^1_m(G_S,\mathbb{G}_m)$$

are isomorphisms.

Proof. That the arrow on the right is an isomorphism was already proven in Theorem 2.22. This result holds even if S is merely assumed to be reduced. For the reader's convenience, we note that the hypotheses of Theorem 2.22 are satisfied due to [Mil17, Cor. 1.32 and 8.39].

We now turn our attention to the sheafification map on the left. As in the proof of Proposition 2.13, we know that G is an extension of an abelian variety A by a linear group L, that is a product of a torus T and a unipotent group U. This leads to the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_S(L,\mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_S(A,\mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_S(G,\mathbb{G}_m) & \longrightarrow & \mathrm{Ext}^1_S(L,\mathbb{G}_m) \\
\| & & \downarrow\wr & & \downarrow & & \downarrow\wr \\
\underline{\mathrm{Hom}}(L,\mathbb{G}_m)(S) & \longrightarrow & \underline{\mathrm{Ext}}^1(A,\mathbb{G}_m)(S) & \longrightarrow & \underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)(S) & \longrightarrow & \underline{\mathrm{Ext}}^1(L,\mathbb{G}_m)(S),
\end{array}$$

whose rows are exact. Propositions 2.14 and 2.25 imply that $\underline{\mathrm{Ext}}^1(L,\mathbb{G}_m)(S)$ vanishes, and Proposition 2.24 further ensures that $\mathrm{Ext}^1_S(L,\mathbb{G}_m)=0$. The desired result then follows from a diagram chase. □

The following example demonstrates that the map $\mathrm{Ext}^1_S(G,\mathbb{G}_m)\to\underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)(S)$ may not be an isomorphism if S is not regular.

■ Example 2.27. Let k be a field, and consider the nodal curve $S=\mathrm{Spec}\,k[x,y]/(y^2-xy-x^3)$. The proof of Proposition 2.19 indicates that $\mathrm{Ext}^1_S(\mathbb{G}_m,\mathbb{G}_m)$ can be computed in the étale topology. Since Grothendieck's proof of Proposition 2.14 also shows that $\underline{\mathrm{Ext}}^1(\mathbb{G}_m,\mathbb{G}_m)$ vanishes on the étale site, Proposition 2.15 gives that

$$\mathrm{Ext}^1_S(\mathbb{G}_m,\mathbb{G}_m)\simeq H^1_{\text{ét}}(S,\mathbb{Z}).$$

As observed in [Wei91, Rem. 5.5.2], the étale cohomology group $H^1_{\text{ét}}(S,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$; implying that the sheafification map $\mathrm{Ext}^1_S(\mathbb{G}_m,\mathbb{G}_m)\to\underline{\mathrm{Ext}}^1(\mathbb{G}_m,\mathbb{G}_m)(S)$ is not an isomorphism. ■

We also prove an analogue of Theorem 2.22 for de Rham spaces.

Theorem 2.28. *Let G be a commutative connected algebraic group over a characteristic zero field k. For a reduced k-scheme S, the natural maps*

$$\underline{\mathrm{Ext}}^1(G_{\mathrm{dR}}, \mathbb{G}_m)(S) \leftarrow \mathrm{Ext}^1_S(G_{\mathrm{dR}}, \mathbb{G}_m) \to H^1_m(G_{\mathrm{dR}} \times S, \mathbb{G}_m)$$

are isomorphisms.

Proof. Proposition 2.13 asserts that the Cartier dual of G_{dR} vanishes. Consequently, Proposition 2.15 implies that the map on the left is an isomorphism for all k-schemes S. To show that the map on the right is also an isomorphism for reduced k-schemes S, we use Lemma 2.21.

Consider a morphism of S-schemes $f\colon G^n_{\mathrm{dR}} \times S \to \mathbb{G}_{m,S}$, where n is either 2 or 3. As in the proof of Theorem 2.22, there exist morphisms $f_i\colon G_S \to \mathbb{G}_{m,S}$ for $i = 1, \ldots, n$, such that their product equals the composition

$$G^n_S \to G^n_{\mathrm{dR}} \times S \to \mathbb{G}_{m,S}.$$

Since epimorphisms in topoi are stable under base change, Proposition A.18 implies that the map $G^n_S \to G^n_{\mathrm{dR}} \times S$ is an epimorphism. Therefore, the morphisms f_i factor through the quotient, yielding maps

$$\bar{f}_i\colon G_{\mathrm{dR}} \times S \to \mathbb{G}_{m,S},$$

whose product is equal to f. □

We will require two further vanishing results. The first concerns extension sheaves of commutative formal groups and is proved in [Rus13, Lem. 1.14]. This result will be applied primarily to the formal completion $\widehat{G}$ of a commutative algebraic group G over a field of characteristic zero, a context in which it first appeared in [BB09, Lem. A.4.6].

Proposition 2.29. *Let $\mathcal{G}$ be a commutative formal group over a field whose Cartier dual is of finite type. Then $\underline{\mathrm{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$ vanishes.*

For the reader's convenience, we provide a proof of this proposition, based on the methods developed in this section, under the assumption that the base field has characteristic zero.

Proof. Let k be the base field of characteristic 0, and set $L := \mathcal{G}^D$. By [Mil17, Prop. 2.37 and Cor. 16.15], there exist a finite étale group scheme F, a torus T, and a vector group U, fitting into a short exact sequence

$$0 \to T \times U \to L \to F \to 0.$$

Dualizing this extension (see [SGA 7_{I}, Exp. VIII, Prop. 3.3.1] and [Stacks, Tag 02KB]) yields a short exact sequence

$$0 \to F^D \to \mathcal{G} \to T^D \times U^D \to 0.$$

Taking the long exact sequence in cohomology associated to $(-)^D = \underline{\mathrm{Hom}}(-, \mathbb{G}_m)$, we reduce to showing that

$$\underline{\mathrm{Ext}}^1(F^D, \mathbb{G}_m) = \underline{\mathrm{Ext}}^1(T^D, \mathbb{G}_m) = \underline{\mathrm{Ext}}^1(U^D, \mathbb{G}_m) = 0.$$

The first vanishing follows from another application of [SGA 7_I, Exp. VIII, Prop. 3.3.1], whereas the second was dealt with in Proposition 2.14. Hence it remains to prove the vanishing for U^D.

Since U is a vector group, after choosing coordinates in U we have that U^D is a power of $\widehat{\mathbb{G}}_a$, so it is enough to show that

$$\mathrm{Ext}^1_R(\widehat{\mathbb{G}}_a, \mathbb{G}_m) = 0$$

for every k-algebra R. By Proposition 2.19, this in turn reduces to proving that $H^1_m(\widehat{\mathbb{G}}_{a,R}, \mathbb{G}_m)$ and $H^2_s(\widehat{\mathbb{G}}_{a,R}, \mathbb{G}_m)$ are trivial. We now establish these two vanishings.

For $n \geqslant 1$, let $\mathbb{G}^{(n)}_{a,R}$ denote the abelian sheaf represented by $\mathrm{Spec}\, R[t]/(t^n)$. By definition, the formal additive group $\widehat{\mathbb{G}}_{a,R}$ is the filtered colimit of the $\mathbb{G}^{(n)}_{a,R}$ in the category of fppf sheaves, and the same colimit computes $\widehat{\mathbb{G}}_{a,R}$ in the $(2,1)$-category of fppf stacks. Consequently, we have a natural equivalence of groupoids

$$\mathrm{Map}(\widehat{\mathbb{G}}_{a,R}, B\mathbb{G}_m) \simeq \mathrm{Map}(\mathrm{colim}_n\, \mathbb{G}^{(n)}_{a,R}, B\mathbb{G}_m) \simeq \lim_n \mathrm{Map}(\mathbb{G}^{(n)}_{a,R}, B\mathbb{G}_m).$$

A $\mathbb{G}_m$-torsor on $\mathbb{G}^{(n)}_{a,R}$ is the same as a line bundle on $\mathrm{Spec}\, R[t]/(t^n)$; i.e. an element of $\mathrm{Pic}(R[t]/(t^n))$. Since the Picard group is invariant under quotienting by nilpotent ideals [Stacks, Tag 0C6R], we have a natural isomorphism $\mathrm{Pic}(R[t]/(t^n)) \simeq \mathrm{Pic}(R)$. Thus every $\mathbb{G}_m$-torsor on $\widehat{\mathbb{G}}_{a,R}$ is pulled back from $\mathrm{Spec}\, R$. Finally, as in the proof of Proposition 2.25, such a torsor is multiplicative precisely if it is trivial. In other words, the group $H^1_m(\widehat{\mathbb{G}}_{a,R}, \mathbb{G}_m)$ vanishes.

Unwinding the definitions, we see that the group $H^2_s(\widehat{\mathbb{G}}_{a,R}, \mathbb{G}_m)$ is the middle cohomology of the complex

$$R[\![x]\!]^\times \xrightarrow{\alpha} R[\![x,y]\!]^\times \xrightarrow{\beta} R[\![x,y,z]\!]^\times \oplus R[\![x,y]\!]^\times,$$

whose differentials are given by

$$\alpha(p(x)) = \frac{p(x+y)}{p(x)p(y)}$$
$$\beta(q(x,y)) = \left(\frac{q(x+y,z)q(x,y)}{q(x,y+z)q(y,z)}, \frac{q(x,y)}{q(y,z)}\right).$$

Let $q(x,y) \in R[\![x,y]\!]^\times$ be a formal power series in the kernel of β. Without loss of generality, we may assume that $q(0,0) = 1$. Since R is a $\mathbb{Q}$-algebra, the usual power series for the exponential and logarithm define mutually inverse group isomorphisms

$$\exp\colon (x_1, \ldots, x_r)R[\![x_1, \ldots, x_r]\!] \rightleftarrows 1 + (x_1, \ldots, x_r)R[\![x_1, \ldots, x_r]\!] :\log.$$

Thus, $g(x,y) := \log(q(x,y)) \in (x,y)R[\![x,y]\!]$ satisfies the additive relations

$$g(x+y,z)+g(x,y) = g(x,y+z)+g(y,z) \quad \text{and} \quad g(x,y) = g(y,x). \tag{$*$}$$

First, setting $y = z = 0$ in ($*$) yields $g(x,0) = g(0,x) = 0$. Next, differentiate the first identity in ($*$) with respect to z. Writing g_i for the partial derivative of g with respect to its i-th variable, we obtain

$$g_2(x+y,z) = g_2(x,y+z)+g_2(y,z).$$

Putting $z = 0$ and using the second relation in ($*$), we get the relations

$$g_1(x,y) = \varphi(x+y)-\varphi(x) \quad \text{and} \quad g_2(x,y) = \varphi(x+y)-\varphi(y),$$

where $\varphi(x) := g_1(0,x) = g_2(x,0)$.

Formally integrating φ, we obtain an element $f(x) \in xR[\![x]\!]$ satisfying $f'(x) = \varphi(x)$. We claim that

$$g(x,y) = f(x+y)-f(x)-f(y).$$

Indeed, the difference $g(x,y)-f(x+y)+f(x)+f(y)$ vanishes at $(0,0)$, and its partial derivatives with respect to both variables are zero. Consequently, the formal power series $p(x) := \exp(f(x)) \in 1+xR[\![x]\!]$ is such that $\alpha(p(x)) = q(x,y)$, finishing the proof. □

Proposition 2.30. *Let A be an abelian variety over a characteristic zero field. Then the abelian sheaves $\underline{\mathrm{Ext}}^2(A,\mathbb{G}_m)$ and $\underline{\mathrm{Ext}}^2(A_{dR},\mathbb{G}_m)$ vanish.*

Proof. The vanishing of $\underline{\mathrm{Ext}}^2(A,\mathbb{G}_m)$ for abelian schemes over an arbitrary base was established in [RR25, Thm. B]. The abelian sheaf $\underline{\mathrm{Ext}}^2(A_{dR},\mathbb{G}_m)$ sits in the exact sequence

$$\underline{\mathrm{Ext}}^1(\widehat{A},\mathbb{G}_m) \to \underline{\mathrm{Ext}}^2(A_{dR},\mathbb{G}_m) \to \underline{\mathrm{Ext}}^2(A,\mathbb{G}_m),$$

and the left-hand term vanishes by the preceding proposition. □

In order to have a bird's-eye view of this section, consider the following definition.

Definition 2.31. Let G be a commutative connected algebraic group over a characteristic zero field k. We denote the abelian sheaf $\underline{\mathrm{Ext}}^1(G_{dR},\mathbb{G}_m)$ by $G^\natural$, and the abelian sheaf $\underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)$ by G'.

Denote by $m\colon G\times G \to G$ the group law of G. For a reduced k-scheme S, Theorem 2.28 gives isomorphisms

$$G^\natural(S) := \underline{\mathrm{Ext}}^1(G_{dR},\mathbb{G}_m)(S) \xleftarrow{\sim} \mathrm{Ext}^1_S(G_{dR},\mathbb{G}_m) \xrightarrow{\sim} H^1_m(G_{dR}\times S,\mathbb{G}_m)$$

that are functorial on G and S. Corollary A.20 then implies that $G^\natural(S)$ is isomorphic to the set of isomorphism classes of line bundles $\mathscr{L}$ on G_S with integrable connection ∇ relative to S satisfying $m^*(\mathscr{L},\nabla) \simeq (\mathscr{L},\nabla) \boxtimes (\mathscr{L},\nabla)$. Moreover, the group structure of $G^\natural(S)$ corresponds to the tensor products of connections.

Similarly, for a regular k-scheme S, Theorem 2.22 yields isomorphisms

$$G'(S) := \underline{\mathrm{Ext}}^1(G,\mathbb{G}_m)(S) \xleftarrow{\sim} \mathrm{Ext}^1_S(G,\mathbb{G}_m) \xrightarrow{\sim} H^1_m(G_S,\mathbb{G}_m)$$

that are functorial on G and S. As above, this implies that $G'(S)$ can be identified to the group of isomorphism classes of line bundles $\mathscr{L}$ on G_S satisfying $m^*\mathscr{L} \simeq \mathscr{L} \boxtimes \mathscr{L}$.

The following remark explains our choice of notation in Definition 2.31.

■ Remark 2.32. Let A be an abelian variety over a characteristic zero field k. According to Proposition 2.14, the abelian sheaf A' in Definition 2.31 is represented by the dual abelian variety. Thus, our notation is not overloaded.

By considering the long exact sequence in cohomology associated with the Cartier duality functor $(-)^D := \underline{\mathrm{Hom}}(-,\mathbb{G}_m)$ and the short exact sequence

$$0 \to \widehat{A} \to A \to A_{dR} \to 0,$$

we deduce that $A^\natural$ is an extension of A' by Ω_A, the vector group of the invariant differentials of A. (Consequently, $A^\natural$ is representable by an algebraic group.[6]) We affirm that $A^\natural$ is the universal vector extension of A' as defined in [MM74, §I.1].

The proof of Theorem 2.28 provides natural isomorphisms

$$A^\natural(S) := \underline{\mathrm{Ext}}^1(A_{dR},\mathbb{G}_m)(S) \xleftarrow{\sim} \mathrm{Ext}^1_S(A_{dR},\mathbb{G}_m) \xrightarrow{\sim} H^1_m(A_{dR}\times S,\mathbb{G}_m)$$

for *all* k-schemes S. This implies that the presheaf $S \mapsto H^1_m(A_{dR}\times S,\mathbb{G}_m)$ already satisfies fppf descent. Therefore, the sheafification in the definition of $E^\natural$ [MM74, Def. I.4.1.6] is superfluous, leading to the identification $A^\natural \simeq E^\natural$.

Mazur and Messing also define an abelian sheaf $\underline{\mathrm{Ext}}^\natural(A,\mathbb{G}_m)$ which, by Remark 2.20, is isomorphic to $A^\natural = \underline{\mathrm{Ext}}^1(A_{dR},\mathbb{G}_m)$. In particular, our methods reprove their [MM74, Prop. I.4.2.1], which compares $\underline{\mathrm{Ext}}^\natural(A,\mathbb{G}_m)$ and $E^\natural$. Finally, from [MM74, Props. I.2.6.7 and I.3.2.3], we conclude that $A^\natural$ is indeed the universal vector extension of A'. ■

3. Moduli of character sheaves

3.1. Generalized 1-motives and their Laumon duals

Let k be a perfect field of any characteristic. In this section, we will systematically apply Proposition 2.3 to represent commutative group stacks over k as two-term complexes (in

[6] We refer the reader to Remark 3.4 for more explanations.

degrees -1 and 0) of abelian sheaves on the site $(\mathsf{Sch}/k)_{\mathrm{fppf}}$. For the reader's convenience, we recall that the Cartier dual of a commutative formal group is represented by an affine commutative group scheme.

Definition 3.1. A *generalized 1-motive* is a two-term complex of abelian sheaves $[\mathcal{G} \to G]$ in $(\mathsf{Sch}/k)_{\mathrm{fppf}}$, where G is a smooth commutative connected algebraic group, and $\mathcal{G}$ is a commutative formal group whose Cartier dual is smooth and connected.

A usual 1-motive, as defined in [Del74, §10.1], is a special case of the definition above where k is algebraically closed, G is a semiabelian variety, and $\mathcal{G}$ is a finitely generated free $\mathbb{Z}$-module. Our Definition 3.1 is inspired by the one in [Rus13] and extends Laumon's [Lau96] to base fields that may have positive characteristic.

Let $[\mathcal{G} \to G]$ be a generalized 1-motive. The Barsotti–Chevalley theorem [Mil17, Thm. 8.27] states that G has a smallest subgroup L such that G/L is proper. This subgroup is affine, smooth and connected, and the quotient G/L is an abelian variety A. In other words, G can be functorially decomposed as an extension

$$0 \to L \to G \to A \to 0,$$

of an abelian variety A by an affine smooth commutative connected algebraic group L.

■ Remark 3.2. Every algebraic group over a characteristic zero field is smooth. In positive characteristic, a singular connected algebraic group is still an extension of an abelian variety by a linear group, but this decomposition may not be unique. See [Bri17a, Ex. 4.3.8]. ■

The motivation for Definition 3.1 arises from the fact that the Cartier duality of Proposition 2.9 naturally restricts to an anti-equivalence of categories

$$\left\{ \begin{array}{c} \text{Affine smooth} \\ \text{commutative connected} \\ \text{algebraic groups over } k \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Commutative formal groups} \\ \text{over } k \text{ whose Cartier dual is} \\ \text{smooth and connected} \end{array} \right\}$$
$$G = \operatorname{Spec} R \longmapsto G^D \simeq \operatorname{Spf} R^*$$
$$\mathcal{G}^D \simeq \operatorname{Spec} R^* \longleftarrow \mathcal{G} = \operatorname{Spf} R.$$

Using this, we will concoct a dual of $[\mathcal{G} \to G]$ of the form $[L^D \to K]$, for some smooth commutative connected algebraic group K fitting into a short exact sequence

$$0 \to \mathcal{G}^D \to K \to A' \to 0,$$

where A' is the dual abelian variety of A. A functorial definition of K is given by the following lemma.

Lemma 3.3. *Let $[\mathscr{G} \to A]$ be the generalized 1-motive defined by the composition $\mathscr{G} \to G \to A$. The complex $\mathrm{R}\underline{\mathrm{Hom}}([\mathscr{G} \to A], \mathbb{G}_m)$ has no cohomology in degrees 0 and 2. Moreover, $\underline{\mathrm{Ext}}^1([\mathscr{G} \to A], \mathbb{G}_m)$ is representable by a smooth commutative connected algebraic group.*

Proof. Applying the functor $\mathrm{R}\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ to the fiber sequence $\mathscr{G} \to A \to [\mathscr{G} \to A]$, we obtain the following long exact sequence:

$$
\begin{array}{l}
0 \longrightarrow \underline{\mathrm{Ext}}^0([\mathscr{G} \to A], \mathbb{G}_m) \longrightarrow 0 \longrightarrow \mathscr{G}^D \\
\quad \longrightarrow \underline{\mathrm{Ext}}^1([\mathscr{G} \to A], \mathbb{G}_m) \longrightarrow A' \longrightarrow 0 \\
\quad \longrightarrow \underline{\mathrm{Ext}}^2([\mathscr{G} \to A], \mathbb{G}_m) \longrightarrow 0.
\end{array}
$$

Here, we use Propositions 2.8, 2.29, and 2.30 to justify the non-trivial zeros above. The vanishing results follow directly, and the representability of $\underline{\mathrm{Ext}}^1([\mathscr{G} \to A], \mathbb{G}_m)$ by a smooth commutative connected algebraic group follows by descent. (See the following remark for an explanation of how these descent arguments work.) □

■ Remark 3.4 — On properties inherited by extensions. Let S be a base scheme, and consider an extension of abelian sheaves on the site $(\mathsf{Sch}/S)_{\mathrm{fppf}}$:

$$0 \to \mathscr{H} \xrightarrow{\varphi} \mathscr{E} \xrightarrow{\psi} \mathscr{G} \to 0.$$

The composition $\mathscr{H} \times \mathscr{E} \xrightarrow{\varphi \times \mathrm{id}} \mathscr{E} \times \mathscr{E} \xrightarrow{m} \mathscr{E}$, where m is the group operation of $\mathscr{E}$, defines an action of $\mathscr{H}$ on $\mathscr{E}$. This action fits into the cartesian diagram

$$
\begin{array}{ccccc}
\mathscr{H} \times \mathscr{E} & \xrightarrow{\varphi \times \mathrm{id}} & \mathscr{E} \times \mathscr{E} & \xrightarrow{m} & \mathscr{E} \\
{\scriptstyle \mathrm{pr}_2} \downarrow & & & & \downarrow {\scriptstyle \psi} \\
\mathscr{E} & & \xrightarrow{\psi} & & \mathscr{G},
\end{array}
$$

giving that $\mathscr{E}$ is an $\mathscr{H}$-torsor over $\mathscr{G}$. In other words, we have an $\mathscr{H}$-equivariant isomorphism $\mathscr{H} \times \mathscr{E} \to \mathscr{E} \times_{\mathscr{G}} \mathscr{E}$.

Assume that $\mathscr{G}$ and $\mathscr{H}$ are represented by group algebraic spaces G and H over S, respectively. By [Stacks, Tag 00WN], there exists a scheme T with a map $T \to \mathscr{E}$ such that the composition

$$T \to \mathscr{E} \to G$$

is an fppf cover. This induces a trivialization $H \times T \simeq \mathscr{E} \times_G T$. Since the definition of an algebraic space is local in the fppf topology [Stacks, Tag 04SK], it follows that $\mathscr{E}$ is representable by an algebraic space E over G. Therefore, E is also a group algebraic space over S.

Let $\mathcal{P}$ be a property of morphisms of algebraic spaces that is stable under base change and fppf-local on the base. The same reasoning implies that if $H \to S$ satisfies $\mathcal{P}$, then so does $E \to G$. Furthermore, if $\mathcal{P}$ is stable under composition and is satisfied by $G \to S$, then $E \to S$ also satisfies $\mathcal{P}$. (See [Stacks, Tags 03H8 and 03YE] for an extensive list of such properties $\mathcal{P}$.)

Now, suppose that S is the spectrum of a field k. If G and H are schemes, then the maps $G \to S$ and $H \to S$ are necessarily separated [Stacks, Tag 047L]. The preceding discussion implies that $E \to S$ is also separated, and so [Stacks, Tag 0B8G] shows that E is a scheme as well.

For the remainder of this discussion, assume that G and H are group algebraic spaces locally of finite type over k. Then, we note that E has dimension $\dim G + \dim H$. Indeed, [Stacks, Tag 0AFH] says that the relative dimension of $E \to G$ is $\dim E - \dim G$. However, this relative dimension is also the dimension of the fiber H, completing the proof.

Since $E \to G$ is faithfully flat of finite presentation, the induced morphism $|E| \to |G|$ between the underlying topological spaces is a quotient map [Stacks, Tag 0413]. Although the fibers of $|E| \to |G|$ may not be homeomorphic to $|H|$, there exists a continuous surjection from $|H|$ to them, as stated in [Stacks, Tag 03H4]. In particular, if G and H are connected, so is E. ∎

Following Laumon, we will systematically denote by K the algebraic group representing $\underline{\mathrm{Ext}}^1([\mathcal{G} \to A], \mathbb{G}_m)$. The following variant of (stacky) Cartier duality was initially introduced in [Del74] and was subsequently generalized in [Lau96] and [Rus13].

Definition 3.5. Let $[\mathcal{G} \to G]$ be a generalized 1-motive. We define the *Laumon dual* $[\mathcal{G} \to G]^{\triangleright}$ to be the generalized 1-motive $[L^D \to K]$, where $L^D \to K$ is the connecting morphism induced by the fiber sequence $L \to [\mathcal{G} \to G] \to [\mathcal{G} \to A]$ via the Cartier duality functor.

We note that the Laumon dual is functorial on the generalized 1-motive. Indeed, by the functoriality of the Barsotti–Chevalley theorem, a morphism of complexes

$$\begin{array}{ccc} \mathcal{G}_1 & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ \mathcal{G}_2 & \longrightarrow & G_2 \end{array}$$

between two generalized 1-motives induces maps $L_1 \to L_2$ and $A_1 \to A_2$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & G_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_2 & \longrightarrow & G_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

commute. Consequently, we obtain a commutative diagram in the derived category of abelian sheaves

$$\begin{array}{ccccc} L_1 & \longrightarrow & [\mathcal{G}_1 \to G_1] & \longrightarrow & [\mathcal{G}_1 \to A_1] \\ \downarrow & & \downarrow & & \downarrow \\ L_2 & \longrightarrow & [\mathcal{G}_2 \to G_2] & \longrightarrow & [\mathcal{G}_2 \to A_2]. \end{array}$$

The Cartier duality functor then induces a morphism $[\mathcal{G}_2 \to G_2]^{\triangleright} \to [\mathcal{G}_1 \to G_1]^{\triangleright}$.

Our next proposition gives a comparison between the Laumon dual defined above and the stacky Cartier dual of Definition 2.5.

Proposition 3.6. *Let $[\mathcal{G} \to G]$ be a generalized 1-motive. There exists a derived category map $[\mathcal{G} \to G]^{\triangleright} \to [\mathcal{G} \to G]^{\vee}$, whose cofiber is $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$.*

Proof. The fiber sequence defining the Laumon dual induces the fiber sequence below.

$$R\underline{\mathrm{Hom}}([\mathcal{G} \to A], \mathbb{G}_m[1]) \to R\underline{\mathrm{Hom}}([\mathcal{G} \to G], \mathbb{G}_m[1]) \to R\underline{\mathrm{Hom}}(L, \mathbb{G}_m[1])$$

By Lemma 3.3, $\underline{\mathrm{Ext}}^2([\mathcal{G} \to A], \mathbb{G}_m)$ vanishes and so $\underline{\mathrm{Ext}}^1([\mathcal{G} \to G], \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$ is an epimorphism. Then, [Bro21, Lem. 3.10] implies that

$$\tau_{\leqslant 0} R\underline{\mathrm{Hom}}([\mathcal{G} \to A], \mathbb{G}_m[1]) \to \tau_{\leqslant 0} R\underline{\mathrm{Hom}}([\mathcal{G} \to G], \mathbb{G}_m[1]) \to \tau_{\leqslant 0} R\underline{\mathrm{Hom}}(L, \mathbb{G}_m[1])$$

is also a fiber sequence. Yet another application of Lemma 3.3 gives that

$$\tau_{\leqslant 0} R\underline{\mathrm{Hom}}([\mathcal{G} \to A], \mathbb{G}_m[1]) \simeq K$$

and so, up to a shift, the fiber sequence just obtained is $L^{\vee}[-1] \to K \to [\mathcal{G} \to G]^{\vee}$. Since $L^D \simeq \tau_{\leqslant 0}(L^{\vee}[-1])$, there is a natural map $L^D \to L^{\vee}[-1]$ making the square

$$\begin{array}{ccccc} L^D & \longrightarrow & K & \longrightarrow & [L^D \to K] \\ \downarrow & & \| & & \\ L^{\vee}[-1] & \longrightarrow & K & \longrightarrow & [\mathcal{G} \to G]^{\vee} \end{array}$$

commute and inducing a morphism of fiber sequences. In this way we obtain the desired comparison map.

Now, according to [Stacks, Tag 08J5], there exists a fiber sequence $L^D \to L^{\vee}[-1] \to \underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$. Finally, the octahedral axiom [Stacks, Tag 05R0] gives that the cofiber of $[\mathcal{G} \to G]^{\triangleright} \to [\mathcal{G} \to G]^{\vee}$ is isomorphic to $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$. □

This proposition implies that the comparison map $[\mathcal{G} \to G]^{\triangleright} \to [\mathcal{G} \to G]^{\vee}$ becomes an isomorphism precisely when $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m) = 0$. This condition is satisfied when G is semiabelian, thereby demonstrating that the Cartier dual on 1-motives, as defined by Deligne, agrees with the stacky Cartier dual. Extending this result, we derive the following corollary.

Corollary 3.7. *The comparison morphism* $[\mathscr{G} \to G]^{\triangleright} \to [\mathscr{G} \to G]^{\vee}$ *is always an isomorphism when the base field* k *has positive characteristic. When* k *has characteristic zero, the comparison map is an isomorphism if and only if* G *is a semiabelian variety.*

Proof. In positive characteristic, the abelian sheaf $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$ always vanishes due to [Ros23, Prop. 2.2.17]. If k has characteristic zero, L is a product of a torus and a vector group U [Mil17, Cor. 16.15]. In particular, Proposition 2.14 implies that $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m) \simeq \underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$. By [RR25, Thm. C], the latter sheaf vanishes precisely when U does. □

For the remainder of this section, we assume the base field k has characteristic zero and $\mathscr{G} = \widehat{G}$, so that $[\widehat{G} \to G] \simeq G_{dR}$. We now provide a more explicit description of $K = \underline{\mathrm{Ext}}^1([\widehat{G} \to A], \mathbb{G}_m)$. The long exact sequence derived from the extension

$$0 \to \widehat{A} \to A \to A_{dR} \to 0,$$

via the Cartier duality functor, results in the short exact sequence

$$0 \to \Omega_A \to A^{\natural} \to A' \to 0,$$

as noted in Remark 2.32. Now, the quotient map $\psi \colon G \to A$ induces a pullback map $\psi^* \colon \Omega_A \to \Omega_G$, and we consider the corresponding pushout extension.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_A & \longrightarrow & A^{\natural} & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi^*} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega_G & \longrightarrow & (\Omega_G \times A^{\natural})/\Omega_A & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

Here, and throughout this work, Ω_G represents the vector group consisting of the *invariant* differentials of G.

Proposition 3.8. *The algebraic group* $K = \underline{\mathrm{Ext}}^1([\widehat{G} \to A], \mathbb{G}_m)$ *is naturally isomorphic to the quotient* $(\Omega_G \times A^{\natural})/\Omega_A$.

Proof. Consider the following commutative diagram, whose rows are fiber sequences.

$$\begin{array}{ccccc} \widehat{G} & \longrightarrow & A & \longrightarrow & [\widehat{G} \to A] \\ \downarrow & & \| & & \downarrow \\ \widehat{A} & \longrightarrow & A & \longrightarrow & [\widehat{A} \to A] \end{array}$$

After applying $R\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ and taking long exact sequences in cohomology, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_A & \longrightarrow & A^\natural & \longrightarrow & A' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \psi^*} & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \Omega_G & \longrightarrow & K & \longrightarrow & A' & \longrightarrow & 0. \end{array}$$

We assert that the square on the left is cocartesian. This means that the complex

$$0 \to \Omega_A \to \Omega_G \times A^\natural \to K \to 0,$$

with the action of Ω_A on $\Omega_G \times A^\natural$ as in the typical pushout construction, is exact. This complex is part of the following larger commutative diagram.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \big\downarrow & & \big\downarrow & & \\ & & 0 & \longrightarrow & \Omega_G & = & \Omega_G & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \Omega_A & \longrightarrow & \Omega_G \times A^\natural & \longrightarrow & K & \longrightarrow & 0 \\ & & \big\| & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \Omega_A & \longrightarrow & A^\natural & \longrightarrow & A' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

In this diagram, every column is exact, and both the top and bottom rows are exact. Therefore, the middle row must also be exact. □

Let us fix some notation. Given that the algebraic group K is a quotient of $\Omega_G \times A^\natural$, we will denote its points as (equivalence classes of) pairs $(\omega, (\mathscr{L}, \nabla))$, where $\omega \in \Omega_G$ and $(\mathscr{L}, \nabla) \in A^\natural$.[7] We will also systematically denote the natural maps $L \to G$ and $G \to A$ as φ and ψ, respectively.

■ Remark 3.9. The morphism $L^D \to K$ in the Laumon dual of $[\widehat{G} \to G]$ can also be explicitly described: it maps a character $\chi \in L^D$ to $[\omega, (\mathscr{L}, \nabla)]$, where ω is any element of Ω_G satisfying $\varphi^*\omega = d\chi/\chi$, and $(\mathscr{L}, \nabla)$ is the unique element of $A^\natural$ satisfying $\psi^*(\mathscr{L}, \nabla) \simeq (\mathcal{O}_G, d - \omega)$. Note that, since the Cartier dual of G_{dR} vanishes, the map $L^D \to K$ is a monomorphism. In particular, the Laumon dual of G_{dR} is a usual abelian sheaf. ■

[7] This is a slight abuse of notation since the sheafification involved in defining the quotient sheaf may be non-trivial. That being said, by Serre vanishing, $K(S)$ indeed is the quotient of $\Omega_G(S) \times A^\natural(S)$ by $\Omega_A(S)$ for affine S.

Finally, we describe the comparison map $[\widehat{G} \to G]^{\triangleright} \to [\widehat{G} \to G]^{\vee}$ from Proposition 3.6. The universal property of pushouts permits us to define the morphism of abelian sheaves

$$\begin{aligned} K &\to G^{\natural} \\ [\omega, (\mathscr{L}, \nabla)] &\mapsto (\mathcal{O}_G, d + \omega) \otimes_{\mathcal{O}_G} \psi^*(\mathscr{L}, \nabla). \end{aligned}$$

This map factors through the quotient, resulting in a morphism $\gamma\colon K/L^D \to G^{\natural}$.

Proposition 3.10. *The morphism $\gamma\colon K/L^D \to G^{\natural}$, sending an equivalence class $[\omega, (\mathscr{L}, \nabla)]$ to the line bundle with integrable connection*

$$(\mathcal{O}_G, d + \omega) \otimes_{\mathcal{O}_G} \psi^*(\mathscr{L}, \nabla),$$

coincides with the comparison map $G_{dR}^{\triangleright} \to G_{dR}^{\vee}$. In particular, γ is a monomorphism and its cokernel is $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$.

Proof. Consider the morphism $K \to G^{\natural}$, defined above using the universal property of pushouts. We affirm that the diagram

$$\begin{array}{ccc} K & \longrightarrow & G^{\natural} \\ \downarrow\wr & & \| \\ \underline{\mathrm{Ext}}^1([\widehat{G} \to A], \mathbb{G}_m) & \longrightarrow & \underline{\mathrm{Ext}}^1([\widehat{G} \to G], \mathbb{G}_m), \end{array}$$

on which the map $\underline{\mathrm{Ext}}^1([\widehat{G} \to A], \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1([\widehat{G} \to G], \mathbb{G}_m)$ is induced by the natural morphism of complexes $[\widehat{G} \to G] \to [\widehat{G} \to A]$, commutes. This is the same as showing that the diagram

$$\begin{array}{ccccc} & & \underline{\mathrm{Ext}}^1([\widehat{A} \to A], \mathbb{G}_m) & & \\ & & \downarrow & \searrow & \\ \underline{\mathrm{Hom}}(\widehat{G}, \mathbb{G}_m) & \longrightarrow & \underline{\mathrm{Ext}}^1([\widehat{G} \to A], \mathbb{G}_m) & & \\ & \searrow & & \searrow & \\ & & & & \underline{\mathrm{Ext}}^1([\widehat{G} \to G], \mathbb{G}_m) \end{array}$$

commutes. The upper triangle clearly commutes by functoriality, and the lower triangle can be seen to commute by applying the functor $R\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ to the morphism of fiber sequences

$$\begin{array}{ccccc} \widehat{G} & \longrightarrow & G & \longrightarrow & [\widehat{G} \to G] \\ \| & & \downarrow \psi & & \downarrow \\ \widehat{G} & \longrightarrow & A & \longrightarrow & [\widehat{G} \to A], \end{array}$$

and taking long exact sequences in cohomology. Now, as in the proof of Proposition 3.6, there are two dashed morphisms making the diagram

$$\begin{array}{ccccc} K & \longrightarrow & [L^D \to K] & \longrightarrow & L^D[1] \\ \| & & \downarrow & & \downarrow \\ K & \longrightarrow & G^\natural & \longrightarrow & L^\vee \end{array}$$

commute: the comparison map of Proposition 3.6 and γ. The fact that they coincide follows from [Stacks, Tag 0FWZ]. □

Recall that the linear part L of G decomposes as a product of a torus T, whose Cartier dual is denoted as X, and a vector group U. Inasmuch as $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m) \simeq \underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$ has no k-points, due to Propositions 2.14 and 2.25, this computation is particularly useful for obtaining concrete information about character sheaves.

Corollary 3.11. *The group of isomorphism classes $H^1_m(G_{dR}, \mathbb{G}_m)$ of character sheaves on G fits into the short exact sequence*

$$0 \to X \to (\Omega_G \times A^\natural(k))/\Omega_A \to H^1_m(G_{dR}, \mathbb{G}_m) \to 0,$$

where the morphism on the left is described in Remark 3.9, and the morphism on the right is as in Proposition 3.10. In particular, every character sheaf on G is of the form

$$(\mathcal{O}_G, d + \omega) \otimes_{\mathcal{O}_G} \psi^*(\mathcal{L}, \nabla),$$

for some $\omega \in \Omega_G$ and $(\mathcal{L}, \nabla) \in A^\natural(k)$.

Proof. The preceding proposition gives an isomorphism $(K/L^D)(k) \simeq H^1_m(G_{dR}, \mathbb{G}_m)$. Now, the group of k-points $(K/L^D)(k)$ fits into the long exact sequence

$$0 \to X \to K(k) \to (K/L^D)(k) \to H^1(k, X) \times H^1(k, \widehat{U^*}),$$

where the term on the right vanishes due to [SGA 7_I, Exp. VIII, Prop. 5.1] and [Bha22, Rem. 2.2.18]. Finally, as noted in Footnote 7, the group $K(k)$ is isomorphic to the quotient $(\Omega_G \times A^\natural(k))/\Omega_A$, completing the proof. □

3.2. The moduli of character sheaves

As discussed in the previous section, Laumon defined a dual $[\widehat{G} \to G]^\triangleright$ that, in a sense, eliminates the enigmatic object $U' = \underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$ from the stacky Cartier dual $G^\natural = G^\vee_{dR}$. Now, according to Remark 3.9, the Laumon dual of G_{dR} is isomorphic to the quotient sheaf $K/(X \times \widehat{U^*})$. Since $\widehat{U^*}$ is a formal group, it cannot be representable. This leads us to the following definition.

Definition 3.12 (Moduli space of character sheaves). We denote by $G^\flat$ the abelian sheaf K/X, where $X \hookrightarrow K$ is the morphism that maps $\chi \in X$ to $[\omega, (\mathscr{L}, \nabla)]$, where ω is any element of Ω_G satisfying $\varphi^*\omega = d\chi/\chi$ and $(\mathscr{L}, \nabla)$ is the unique element of $A^\natural$ satisfying $\psi^*(\mathscr{L}, \nabla) \simeq (\mathcal{O}_G, d - \omega)$.

Note that although the map $X \to K$ is a monomorphism, it is not an immersion. Consequently, it is unclear whether the quotient $G^\flat = K/X$ is a scheme, and it may indeed fail to be one. A brief verification shows that the morphism

$$\begin{aligned} k &\to H^1_m(\mathbb{G}_{m,dR}, \mathbb{G}_m) \\ \alpha &\mapsto (\mathcal{O}_{\mathbb{G}_m}, d - \alpha\, dx/x) \end{aligned}$$

is surjective and induces an isomorphism $H^1_m(\mathbb{G}_{m,dR}, \mathbb{G}_m) \simeq k/\mathbb{Z}$. Accordingly, the moduli space $\mathbb{G}_m^\flat$ is isomorphic to $\mathbb{G}_a/\mathbb{Z}$, an algebraic space[8] that is not a scheme. Nonetheless, the main result of this section, Theorem 3.16, asserts in Corollary 3.17 that $G^\flat$ is as well-behaved as one could hope for.

■ Remark 3.13. Let S be a k-scheme. According to Propositions 2.13, 2.15, and 3.6, there are natural morphisms

$$G^\flat(S) \xrightarrow{(1)} G^{\triangleright}_{dR}(S) \xrightarrow{(2)} G^\natural(S) \xrightarrow{(3)} H^1_m(G_{dR} \times S, \mathbb{G}_m).$$

Proposition 2.24 and Theorem 2.28 imply that the maps (1) and (3) are isomorphisms if S is reduced. Furthermore, Proposition 2.25 states that (2) is an isomorphism for seminormal k-schemes S. In particular, $G^\flat(k)$ is isomorphic to the group of isomorphism classes of character sheaves on G, thereby justifying its name. ■

■ Remark 3.14. The assignment $G \mapsto G^\flat$ is functorial. Specifically, consider a morphism $G_1 \to G_2$ between commutative connected algebraic groups over a characteristic zero field. Since the Barsotti–Chevalley decomposition is functorial, we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & G_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_2 & \longrightarrow & G_2 & \longrightarrow & A_2 & \longrightarrow & 0, \end{array}$$

where A_i are abelian varieties and L_i are affine. The linear parts L_i further decompose into a product of tori T_i and unipotent groups U_i. By [Mil17, Cor. 14.18], the morphism $L_1 \to L_2$ is a product of morphisms $T_1 \to T_2$ and $U_1 \to U_2$. In particular, this induces a

[8]We emphasize that, as in [Stacks, Tag 025Y] and contrarily to [LM00, Déf. 1.1], we *do not* suppose that algebraic spaces are quasi-separated.

map $X_2 \to X_1$ between the character groups of T_2 and T_1, respectively. The functoriality of the Laumon dual ensures that the square on the right of the diagram

$$\begin{array}{ccccc} X_2 & \longrightarrow & X_2 \times \widehat{U_2^*} & \longrightarrow & K_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_1 \times \widehat{U_1^*} & \longrightarrow & K_1 \end{array}$$

commutes. Consequently, the universal property of cokernels gives the desired map $G_2^\flat \to G_1^\flat$. ∎

∎ Example 3.15. As usual, let T be a torus, U a unipotent group, and A be an abelian variety. We have that

$$T^\flat \simeq T^\natural \simeq \Omega_T/X \simeq t^*/X, \quad U^\flat \simeq \Omega_U \simeq U^*, \quad A^\flat \simeq A^\natural.$$

The first one is a group algebraic space, while the other two are algebraic groups. ∎

Before going further, we note that there exists a large diagram relating many of the objects appearing in this section. Since both formal completions and the de Rham functor are exact, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{T} \times \widehat{U} & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T \times U & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_{dR} \times U_{dR} & \longrightarrow & G_{dR} & \longrightarrow & A_{dR} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

in which every column and row is exact. By applying the Cartier duality functor, and passing to the long exact sequences in cohomology, we obtain

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & G^{D} & \xrightarrow{\varphi^*} & X \times \widehat{U^*} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_A & \xrightarrow{\psi^*} & \Omega_G & \xrightarrow{\varphi^*} & \Omega_T \times \Omega_U & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{\natural} & \xrightarrow{\psi^*} & G^{\natural} & \xrightarrow{\varphi^*} & T^{\natural} \times U^{\natural} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & A' & \xrightarrow{\psi^*} & G' & \xrightarrow{\varphi^*} & U' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Here, every row (including the zigzag path) and every column is exact. The necessary computations and vanishing results have already been discussed in the previous section. We note that the morphisms in the columns have natural geometric interpretations, as given by

$$
\begin{array}{ccccccccc}
0 \longrightarrow & G^{D} & \longrightarrow & \Omega_G & \longrightarrow & G^{\natural} & \longrightarrow & G' & \longrightarrow 0 \\
 & \chi & \longmapsto & d\chi/\chi & & (\mathscr{L}, \nabla) & \longmapsto & \mathscr{L} & \\
 & & & \omega & \longmapsto & (\mathcal{O}_G, d + \omega). & & &
\end{array}
$$

We are now in position to state the main theorem of this section.

Theorem 3.16. *Let* G *be a commutative connected algebraic group over a characteristic zero field. Write* G *as an extension of an abelian variety* A *by a product* $T \times U$ *of a torus* T *and a unipotent group* U*. Then the complex* $0 \to A^{\flat} \to G^{\flat} \to T^{\flat} \times U^{\flat} \to 0$ *is exact.*

Proof. Consider the map $K \to \Omega_T \times \Omega_U$ induced by $\varphi^* \colon \Omega_G \to \Omega_T \times \Omega_U$ and $0 \colon A^{\natural} \to \Omega_T \times \Omega_U$. We claim that the composition $K \to \Omega_T \times \Omega_U \to \Omega_T/X \times \Omega_U$ descends to the quotient K/X. According to the universal property of the quotient, we need to verify that the composition

$$X \to X \times \widehat{U^*} \to K \to \Omega_T \times \Omega_U \to \Omega_T/X \times \Omega_U$$

is zero. Applying the functor $\mathrm{R}\underline{\mathrm{Hom}}(-,\mathbb{G}_m)$ to the morphism of fiber sequences

$$\begin{array}{ccccc} L & \longrightarrow & [\widehat{L} \to L] & \longrightarrow & \widehat{L}[1] \\ \| & & \downarrow & & \downarrow \\ L & \longrightarrow & [\widehat{G} \to G] & \longrightarrow & [\widehat{G} \to A], \end{array}$$

and taking long exact sequences in cohomology, we find that the composition $X \times \widehat{U^*} \to K \to \Omega_T \times \Omega_U$ is the familiar map appearing on Page 35. In particular, this composition is the product of $X \to \Omega_T$ and $\widehat{U^*} \to \Omega_U$. It follows that our large composition vanishes, and we obtain a map $K/X \to \Omega_T/X \times \Omega_U$.

Now, we have every morphism needed to consider the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & X & = & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^\natural & \longrightarrow & K & \longrightarrow & \Omega_T \times \Omega_U & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^\natural & \longrightarrow & K/X & \longrightarrow & \Omega_T/X \times \Omega_U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

whose columns are clearly exact. Since the top row is also exact, by the nine-lemma, it suffices to prove that the middle row is exact. This holds by an application of the snake lemma in the pushout extension defining K. □

The theorem above implies that $G^\flat$ is a "coarse moduli space" for $G^\natural$, in the sense that $G^\flat$ is represented by an algebraic space with the same k-points as $G^\natural$.[9]

Corollary 3.17. *The abelian sheaf* $G^\flat$ *is representable by a smooth commutative connected group algebraic space. Moreover, it satisfies* $\dim G \leqslant \dim G^\flat \leqslant 2 \dim G$, *with equality on the left if and only if* G *is affine, and equality on the right if and only if* G *is proper.*

Proof. The discussion in Remark 3.4 indicates that $G^\flat$ is representable by a smooth commutative connected group algebraic space of dimension $\dim T^\flat + \dim U^\flat + \dim A^\flat$.

[9] However, it remains unclear whether there is a natural morphism $G^\natural \to G^\flat$, or if such a morphism would satisfy the universal property.

Given that

$$\begin{aligned}\dim T^\flat &= \dim \Omega_T/X = \dim \Omega_T = \dim T\\ \dim U^\flat &= \dim U^* = \dim U\\ \dim A^\flat &= \dim A^\natural = \dim \Omega_A + \dim A' = 2\dim A,\end{aligned}$$

we find that $\dim G^\flat = \dim T + \dim U + 2\dim A = \dim G + \dim A$, thus completing the proof. □

The full subcategory of $\mathsf{Ab}((\mathsf{Sch}/k)_{\mathrm{fppf}})$ consisting of commutative connected algebraic groups is closed under extensions. Consequently, it inherits a structure of exact category. One might believe that this makes the functor $G \mapsto G^\flat$ exact. The following results provide some support for this conjecture.

Proposition 3.18. *Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence of abelian varieties over a characteristic zero field. Then the complex $0 \to A_3^\flat \to A_2^\flat \to A_1^\flat \to 0$ is exact.*

Proof. By Proposition A.18, the complex $0 \to A_{1,\mathrm{dR}} \to A_{2,\mathrm{dR}} \to A_{3,\mathrm{dR}} \to 0$ is exact. The Cartier duality functor then induces the exact sequence

$$\underline{\mathrm{Hom}}(A_{1,\mathrm{dR}}, \mathbb{G}_m) = 0 \to A_3^\natural \to A_2^\natural \to A_1^\natural \to 0 = \underline{\mathrm{Ext}}^2(A_{3,\mathrm{dR}}, \mathbb{G}_m),$$

where the vanishing results are due to Propositions 2.13 and 2.30. □

Proposition 3.19. *Let $0 \to L_1 \to L_2 \to L_3 \to 0$ be a short exact sequence of linear commutative connected algebraic groups over a characteristic zero field. Then the complex $0 \to L_3^\flat \to L_2^\flat \to L_1^\flat \to 0$ is exact.*

Proof. First, we decompose the linear groups L_i as a product of tori T_i and unipotent groups U_i. The exactness of the formal completion functor induces the short exact sequence

$$0 \to \widehat{T}_1 \times \widehat{U}_1 \to \widehat{T}_2 \times \widehat{U}_2 \to \widehat{T}_3 \times \widehat{U}_3 \to 0.$$

Then, the Cartier duality functor gives rise to the exact sequence

$$0 \to \Omega_{T_3} \times \Omega_{U_3} \to \Omega_{T_2} \times \Omega_{U_2} \to \Omega_{T_1} \times \Omega_{U_1} \to 0 = \underline{\mathrm{Ext}}^1(\widehat{T}_3 \times \widehat{U}_3, \mathbb{G}_m).$$

Here, we use Corollary 2.12 and Proposition 2.29. As in Remark 3.14, we have induced maps $X_3 \to X_2 \to X_1$ between the character groups of the tori T_i, fitting into the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{T_3} \times \Omega_{U_3} & \longrightarrow & \Omega_{T_2} \times \Omega_{U_2} & \longrightarrow & \Omega_{T_1} \times \Omega_{U_1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{T_3}/X_3 \times \Omega_{U_3} & \longrightarrow & \Omega_{T_2}/X_2 \times \Omega_{U_2} & \longrightarrow & \Omega_{T_1}/X_1 \times \Omega_{U_1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

The first row above is exact due to Proposition 2.14, and the nine lemma implies that the bottom row is also exact. □

PROPOSITION 3.20. *Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be a short exact sequence of commutative connected algebraic groups over a characteristic zero field. Then $G_3^\flat \to G_2^\flat$ is a monomorphism and $G_2^\flat \to G_1^\flat$ is an epimorphism.*

Proof. Consider the following commutative diagram, composed of the short exact sequence of commutative connected algebraic groups G_i, along with the induced maps on their Barsotti–Chevalley decompositions.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_1 \times U_1 & \longrightarrow & T_2 \times U_2 & \longrightarrow & T_3 \times U_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The exactness of Barsotti–Chevalley decompositions has been thoroughly examined by Brion in [Bri17b]. He proved that the complex $0 \to U_1 \to U_2 \to U_3 \to 0$ is exact, but

$$0 \to T_1 \to T_2 \to T_3 \to 0 \qquad \text{and} \qquad 0 \to A_1 \to A_2 \to A_3 \to 0$$

might only be exact up to isogeny [Bri17b, Thm. 2.9, Lems. 4.3 and 4.7]. That being said, the map $T_1 \to T_2$ is a monomorphism, while $T_2 \to T_3$ and $A_2 \to A_3$ are epimorphisms.[10]

Next, we apply the functor $(-)^\flat$ to the diagram above, obtaining the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_3^\flat & \longrightarrow & A_2^\flat & \longrightarrow & A_1^\flat & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G_3^\flat & \longrightarrow & G_2^\flat & \longrightarrow & G_1^\flat & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_3^\flat \times U_3^\flat & \longrightarrow & T_2^\flat \times U_2^\flat & \longrightarrow & T_1^\flat \times U_1^\flat & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

whose rows are complexes and columns are exact. According to the preceding propositions, the maps $A_3^\flat \to A_2^\flat$ and $T_3^\flat \times U_3^\flat \to T_2^\flat \times U_2^\flat$ are monomorphisms. The long exact sequence in cohomology associated with the short exact sequence of complexes above implies that $G_3^\flat \to G_2^\flat$ is also a monomorphism.

We claim that $A_2^\flat \to A_1^\flat$ is an epimorphism. Let F be the kernel of $A_1 \to A_2$ and B be its cokernel. Note that F is a finite group, while B is an abelian variety. Consider the short exact sequences

$$0 \to F \to A_1 \to A_1/F \to 0 \qquad \text{and} \qquad 0 \to A_1/F \to A_2 \to B \to 0.$$

By passing to the de Rham spaces and considering the long exact sequences induced by the Cartier duality functor, we obtain that the maps $A_2^\flat \to (A_1/F)^\flat$ and $(A_1/F)^\flat \to A_1^\flat$ are epimorphisms. Hence, so is their composition. Since $T_2^\flat \times U_2^\flat \to T_1^\flat \times U_1^\flat$ is an epimorphism, the same argument as above proves the same for $G_2^\flat \to G_1^\flat$. □

As it turns out, the lack of exactness in the Barsotti–Chevalley decompositions is more than just a nuisance in the proof above. Inspired by [Bri17b, Rem. 4.6], we provide below a counter-example to our conjecture.

■ Example 3.21. Assume the base field k contains a non-trivial p-th root of unity, for some prime p. Given an elliptic curve E containing a rational point of order p, we let μ_p act

[10] Brion's results apply in arbitrary characteristic, while the characteristic zero case required here is significantly simpler.

diagonally on the product $E \times \mathbb{G}_m$ and consider the quotient G. This group fits into the short exact sequence

$$0 \to E \to G \to \mathbb{G}_m \to 0.$$

Taking the Barsotti–Chevalley decomposition of each group, we obtain the following commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & \longrightarrow & \mathbb{G}_m & = & \mathbb{G}_m & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \| & & \\
 & & 0 & \longrightarrow & E & \longrightarrow & G & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_p & \longrightarrow & E & \longrightarrow & E/\mu_p & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0, & & & &
\end{array}$$

whose rows and columns are exact. Passing to the de Rham spaces on the bottom row and taking the long exact sequence in cohomology induced by the Cartier dual, we obtain the exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to (E/\mu_p)^\flat \to E^\flat \to 0.$$

The functor $(-)^\flat$ can be applied to the portion of the preceding diagram consisting of commutative connected algebraic groups. This yields the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & (E/\mu_p)^\flat & \longrightarrow & E^\flat & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{G}_m^\flat & \longrightarrow & G^\flat & \longrightarrow & E^\flat & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{G}_m^\flat & = & \mathbb{G}_m^\flat & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array}$$

where the columns are exact and the rows are complexes. The long exact sequence in cohomology induced from a short exact sequence of complexes then identifies the middle cohomology of

$$0 \to \mathbb{G}_m^\flat \to G^\flat \to E^\flat \to 0$$

with $\mathbb{Z}/p\mathbb{Z}$, thereby proving that the functor $(-)^\flat$ is not exact. ∎

3.3. Linear and generic subspaces of the moduli space

For an abelian variety A, certain *big* subsets of $A^\flat$ play a central role in non-abelian Hodge theory [Sim93] and in the formulation of generic vanishing theorems for de Rham cohomology [Sch15]. For a torus T, similar *big* subsets of $T^\flat$ were also studied in [Sab92]. In this section, we extend the definition of these subsets—see Definition 3.25—to arbitrary commutative connected algebraic groups.

Definition 3.22 (Linear subspace). Let G be a commutative connected algebraic group over a characteristic zero field. For an epimorphism $\rho\colon G \twoheadrightarrow \widetilde{G}$ with connected kernel, the image of $\rho^\flat\colon \widetilde{G}^\flat \hookrightarrow G^\flat$ is said to be a *linear subspace* of $G^\flat$.

The following remark concretely characterizes linear subspaces of $G^\flat$ when G is affine. Furthermore, when G is an abelian variety, this notion is related to existing concepts in the literature.

∎ Remark 3.23. Let $G \twoheadrightarrow \widetilde{G}$ be an epimorphism between commutative connected algebraic groups with connected kernel N. If G is linear, unipotent, a torus, or an abelian variety, then $\widetilde{G}$ and N inherit the same properties. The following characterizations follow from this observation.

Consider a torus T with character group X. A linear subspace of $T^\flat \simeq \Omega_T/X$ is of the form V/Y, where Y is a subgroup of X and V is the linear subspace of Ω_T generated by Y, in the sense of linear algebra. For a unipotent group U, a linear subspace of $U^\flat = U^*$ corresponds to a linear subspace of the underlying vector space of U^*, in the sense of linear algebra.

Let L be a linear commutative connected algebraic group, and decompose it as a product of a torus T and a unipotent group U. If $\rho\colon L \twoheadrightarrow \widetilde{L}$ is an epimorphism with connected kernel, then $\widetilde{L}$ is also linear and decomposes as $\widetilde{L} \simeq \widetilde{T} \times \widetilde{U}$. Moreover, ρ is a product of epimorphisms $T \twoheadrightarrow \widetilde{T}$ and $U \twoheadrightarrow \widetilde{U}$. Thus, a linear subspace of $L^\flat$ is the product of linear subspaces of $T^\flat$ and $U^\flat$.

For an abelian variety A, linear subspaces of $A^\flat \simeq A^\natural$ were first studied by Simpson, who termed them *triple tori* [Sim93, p. 365]. Schnell refers to translates of linear subspaces of $A^\flat$ as *linear subvarieties* [Sch15, Def. 2.3]. ∎

Proposition 3.24. *Let G be a commutative connected algebraic group over a characteristic zero field, and express G as an extension of an abelian variety A by a linear group L. If $V \subset L^\flat$ is a (translate of a) linear subspace, so is its inverse image by $G^\flat \to L^\flat$.*

Proof. Let $L \twoheadrightarrow \widetilde{L}$ be the epimorphism defining the linear subspace V, and let N be its kernel. Denote by $\widetilde{G}$ the quotient of G by N; we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & \widetilde{L} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \widetilde{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & A & = & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

whose rows and columns are exact. Applying the functor $(-)^\flat$, we obtain the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A^\flat & = & A^\flat & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{G}^\flat & \longrightarrow & G^\flat & \longrightarrow & N^\flat & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \widetilde{L}^\flat & \longrightarrow & L^\flat & \longrightarrow & N^\flat & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

According to Theorem 3.16 and Proposition 3.19, its columns are exact, as well as the top and bottom rows. Consequently, the middle row is exact as well. A diagram chase shows that the square

$$\begin{array}{ccc}
\widetilde{G}^\flat & \longrightarrow & G^\flat \\
\downarrow & & \downarrow \\
\widetilde{L}^\flat & \longrightarrow & L^\flat
\end{array}$$

is cartesian, thereby concluding the proof. □

Definition 3.25 (Generic subspace). Let G be a commutative connected algebraic group over a characteristic zero field. A *generic subspace* of $G^\flat$ is the complement of a finite union of translates of linear subspaces of $G^\flat$ with positive codimension.

In algebraic geometry, a property is often said to hold generically if it holds on an open dense subset. For extensions of abelian varieties by unipotent groups, this can be related with our definition above.

Proposition 3.26. *Let* G *be a commutative algebraic group over a characteristic zero field that is an extension of an abelian variety by a unipotent group. If* V *is a generic subspace of* $G^\flat$*, then* V *is open and dense in* $G^\flat$*.*

Proof. Since the intersection of a finite number of open dense subsets is also open and dense, it suffices to prove that the complement of a linear subspace with positive codimension is open and dense. Let $G \twoheadrightarrow \widetilde{G}$ be an epimorphism with connected kernel N defining the linear subspace $\widetilde{G}^\flat$ of $G^\flat$.

First, we claim that $\widetilde{G}$ is also an extension of an abelian variety by a unipotent group. Denote by U the maximal unipotent subgroup of G and by A the quotient G/U. The universal property of quotients yields a morphism $N/(U \cap N) \to A$ making the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & U \cap N & \longrightarrow & N & \longrightarrow & N/(U \cap N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

commute. A quick diagram chase shows that $N/(U \cap N) \to A$ is a monomorphism. The usual isomorphism theorems then imply that $(G/N)/(U/(U \cap N))$ is isomorphic to $(G/U)/(N/(U \cap N))$. Denoting by B this common group, we may complete the diagram above to

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U \cap N & \longrightarrow & N & \longrightarrow & N/(U \cap N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U/(U \cap N) & \longrightarrow & \widetilde{G} & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Here every column and every row is exact. Since B is a quotient of an abelian variety, it is also an abelian variety. Similarly, $U/(U \cap N)$ is a unipotent group.

Recall from Corollary 3.17 that $G^\flat$ and $\widetilde{G}^\flat$ are commutative connected algebraic groups. Since algebraic groups in characteristic zero are smooth, $G^\flat$ is irreducible. According to [Mil17, Thm. 5.34], the map $\widetilde{G}^\flat \to G^\flat$ is a closed immersion and so its complement is open. Since $\widetilde{G}^\flat$ is assumed to be of positive codimension, its complement is non-empty. The irreducibility of $G^\flat$ then implies that it is dense, thus completing the proof. □

The preceding result does not hold for general commutative connected algebraic groups. The Zariski topology of $\mathbb{G}_m^\flat \simeq \mathbb{A}^1/\mathbb{Z}$ is too coarse. Indeed, the only open dense subset of $\mathbb{G}_m^\flat$ is the entire space itself. This suggests the following definition.

Definition 3.27 (Analytic moduli space). Let G be a commutative connected algebraic group over $\mathbb{C}$. Taking the $\mathbb{C}$-points of the morphism $X \hookrightarrow K$ defining the moduli space $G^\flat$, we obtain a monomorphism of abelian groups $X \hookrightarrow K(\mathbb{C})$. Equip X with the discrete topology and $K(\mathbb{C})$ with the analytic topology. The quotient $K(\mathbb{C})/X$ is said to be the *analytic moduli space* and is denoted by $G^\flat_{an}$.

Interestingly, although $G^\flat$ is often not representable by a scheme, $G^\flat_{an}$ is a complex manifold. Furthermore, for an abelian variety A, the complex manifold $A^\flat_{an}$ is Stein, even though $A^\flat$ is not affine. The moduli space $A^\flat$ is even *anti-affine*, meaning that every morphism of algebraic varieties $A^\flat \to \mathbb{A}^1$ is constant [Bri17a, Prop. 5.5.8].

For a generic subspace V of $G^\flat$, we denote by V_{an} the set $V(\mathbb{C})$ with the subspace topology inherited from $G^\flat_{an}$. An analytic analogue of Proposition 3.26 holds in all generality.

Proposition 3.28. *Let G be a commutative connected algebraic group over $\mathbb{C}$. If V is a generic subspace of $G^\flat$, then V_{an} is an open subset of $G^\flat_{an}$ whose complement has measure zero. In particular, it is an open dense subset.*

Proof. Let $G \twoheadrightarrow \widetilde{G}$ be an epimorphism with connected kernel N defining a linear subspace $\widetilde{G}^\flat$ of $G^\flat$ with positive codimension. As in the proof of Proposition 3.26, it suffices to prove that $\widetilde{G}^\flat_{an}$ is a closed subset of $G^\flat_{an}$ with measure zero.

According to the functoriality of $(-)^\flat$, explained in Remark 3.14, the natural map $\widetilde{G}^\flat_{an} \hookrightarrow G^\flat_{an}$ is induced by a morphism of schemes $\widetilde{K} \to K$. Consequently, $\widetilde{G}^\flat_{an}$ is a complex Lie subgroup of $G^\flat_{an}$. It follows that $\widetilde{G}^\flat_{an} \hookrightarrow G^\flat_{an}$ is a closed immersion [Bou06, Chap. III, §1.3.5]. Since we assumed that $\dim \widetilde{G}^\flat < \dim G^\flat$, Sard's theorem implies that $\widetilde{G}^\flat_{an}$ has measure zero in $G^\flat_{an}$. □

Definition 3.29 (Character variety). Let G be a commutative connected algebraic group over $\mathbb{C}$. The *character variety* of G, denoted $\mathrm{Char}(G)$, is the spectrum of the group algebra $\mathbb{C}[\pi_1(G_{an})]$.

Recall that the group algebra functor $\mathsf{Grp} \to \mathsf{Alg}(\mathbb{C})$ is left adjoint to the functor sending a $\mathbb{C}$-algebra R to its group of units $R^\times$. Thus, the $\mathbb{C}$-points of $\mathrm{Char}(G)$ correspond to characters $\pi_1(G_{an}) \to \mathbb{C}^\times$ of the fundamental group $\pi_1(G_{an})$. In other words, $\mathrm{Char}(G)$

is a moduli space for rank one local systems on G_{an}. The Hopf algebra structure in $\mathbb{C}[\pi_1(G_{an})]$ induces a structure of algebraic group in $\mathrm{Char}(G)$, encoding the tensor product of local systems.

The Riemann–Hilbert equivalence states that local systems on G_{an} correspond to *regular* connections on G. Since all non-trivial character sheaves on unipotent groups are irregular, $\mathrm{Char}(G)$ cannot parametrize character sheaves in general. On the other hand, character sheaves on semiabelian varieties are regular, allowing such a comparison.

PROPOSITION 3.30. *Let* G *be a semiabelian variety over* $\mathbb{C}$. *Then the complex Lie groups* $G^\flat_{an}$ *and* $\mathrm{Char}(G)_{an}$ *are isomorphic.*

Proof. Write G as an extension of an abelian variety A by a torus T. By analytification, we obtain a short exact sequence

$$0 \to T_{an} \to G_{an} \to A_{an} \to 0$$

of complex Lie groups. In particular, $G_{an} \to A_{an}$ is a fibration with fiber T_{an}. Considering the associated long exact sequence in homotopy, we have

$$0 = \pi_2(A_{an}) \to \pi_1(T_{an}) \to \pi_1(G_{an}) \to \pi_1(A_{an}) \to \pi_0(T_{an}) = 0.$$

Here, $\pi_0(T_{an})$ vanishes since T_{an} is connected, and $\pi_2(A_{an})$ vanishes since A_{an} is a Lie group. This induces a short exact sequence of Hopf algebras

$$\mathbb{C} \to \mathbb{C}[\pi_1(T_{an})] \to \mathbb{C}[\pi_1(G_{an})] \to \mathbb{C}[\pi_1(A_{an})] \to \mathbb{C},$$

followed by a short exact sequence of affine algebraic groups

$$0 \to \mathrm{Char}(A) \to \mathrm{Char}(G) \to \mathrm{Char}(T) \to 0.$$

There is a natural holomorphic morphism $G^\flat_{an} \to \mathrm{Char}(G)_{an}$ sending a character sheaf $(\mathscr{L}, \nabla)$ to the local system $\ker \nabla_{an}$. This is compatible with inverse images and tensor products, giving rise to the following commutative diagram of complex Lie groups.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^\flat_{an} & \longrightarrow & G^\flat_{an} & \longrightarrow & T^\flat_{an} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Char}(A)_{an} & \longrightarrow & \mathrm{Char}(G)_{an} & \longrightarrow & \mathrm{Char}(T)_{an} & \longrightarrow & 0
\end{array}$$

Since character sheaves on A and T are regular, the Riemann–Hilbert correspondence implies that the morphisms $A^\flat_{an} \to \mathrm{Char}(A)_{an}$ and $T^\flat_{an} \to \mathrm{Char}(T)_{an}$ are injective. The fact that a line bundle with integrable connection on A is automatically a character sheaf further says that $A^\flat_{an} \to \mathrm{Char}(A)_{an}$ is surjective.

To prove that $T^\flat_{an} \to \mathrm{Char}(T)_{an}$ is surjective, we may suppose that $T = \mathbb{G}_m$. A line bundle with integrable connection on $\mathbb{G}_m$ is of the form $(\mathcal{O}_{\mathbb{G}_m}, d + f(x)\,dx/x)$, for some $f \in k[x, x^{-1}]$. This connection is regular precisely if f is constant. In other words, regular line bundles with integrable connection on $\mathbb{G}_m$ are the same as character sheaves. This proves that $(\mathbb{G}_m)^\flat_{an} \to \mathrm{Char}(\mathbb{G}_m)_{an}$ is surjective. □

COROLLARY 3.31. *Let* G *be a semiabelian variety over* $\mathbb{C}$. *Then every character sheaf on* G *is regular.*

A. CONNECTIONS AND DE RHAM SPACES

As it was first observed by Simpson [Sim96], given a choice of cohomology theory H and a "space" X, there is often a stack X_H whose category of quasi-coherent sheaves coincides with the category of coefficients for H. Moreover, the association $X \mapsto X_H$ preserves the functoriality of the given cohomology theory.

In this appendix, we study the de Rham side of this story. Namely, given a variety X, the *de Rham space* X_{dR} has the marvellous property that $\mathbb{G}_m$-torsors over it are the same as line bundles on X *endowed with a flat connection*. Moreover, formal completions can also be understood in function of the de Rham spaces.

The author claims no originality for any result in this appendix: all the results in it are either available in the literature or are folklore. (See [GR17] and [Hen17] for more on this.) However, even the results that have published proofs are usually studied in the context of derived algebraic geometry, so we thought that this appendix could be helpful to some readers.

A.1. BASIC PROPERTIES OF THE DE RHAM SPACE

Let k be a field and consider the category Aff/k of affine schemes over k. In order to simplify notation, we will often denote an object $\mathrm{Spec}\, R$ of Aff/k as R.

DEFINITION A.1 (de Rham space). Given a presheaf of sets X on Aff/k, its *de Rham space* X_{dR} is the presheaf defined by

$$X_{dR}(R) := \operatorname*{colim}_{I \subset R} X(R/I),$$

where the colimit runs over the filtered poset of nilpotent ideals of R. This presheaf comes equipped with a morphism $X \to X_{dR}$ induced by the trivial ideal $I = 0$.

This assignment is functorial: given a morphism of presheaves $f\colon X \to Y$, there is an induced map $f_{dR}\colon X_{dR} \to Y_{dR}$, making the diagram

$$\begin{array}{ccc} X & \longrightarrow & X_{dR} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{dR}} \\ Y & \longrightarrow & Y_{dR} \end{array}$$

commute. As it will be formalized in Corollary A.7, the geometric interpretation of X_{dR}, at least for smooth k-schemes X, is that it is a quotient of X where we identify infinitesimally close points.

We begin our study of the de Rham space with the following simple observation, which will prove useful later.

Proposition A.2. *The functor* $(-)_{dR}\colon \mathsf{PSh}(\mathsf{Aff}/k) \to \mathsf{PSh}(\mathsf{Aff}/k)$ *preserves finite limits and arbitrary colimits.*

Proof. Since limits and colimits of presheaves are computed pointwise, this follows from the fact that filtered colimits in the category of sets commute with arbitrary colimits and finite limits. □

For a finite type k-algebra R (or more generally, a noetherian k-algebra), the nilradical $\mathrm{Nil}(R)$ is nilpotent as it is generated by finitely many nilpotent elements. Consequently, $X_{dR}(R) \simeq X(R/\mathrm{Nil}(R)) = X(R_{red})$. This property holds for any k-algebra provided that X is a locally of finite type k-scheme.

Proposition A.3. *Let* X *be a locally of finite type scheme over* k*. Then* $X_{dR}(R) \simeq X(R_{red})$ *for every* k*-algebra* R*.*

Proof. Define $S := \mathrm{colim}_{I\subset R} R/I$, where the colimit runs through the nilpotent ideals of R. As usual, elements of S are denoted as equivalence classes of the form $[I, x]$, where I is a nilpotent ideal in R and x is an element of R. Here, $[I, x] = [I', x']$ if there exists a nilpotent ideal J containing I and I' such that $x \equiv x' \mod J$.

The natural map $R \to S$, corresponding to the ideal $I = 0$, sends every nilpotent in R to zero. In other words, it factors through the nilradical yielding a map $R_{red} \to S$. We affirm that this morphism is injective. Indeed, $[0, x] = 0$ means that there exists a nilpotent ideal J containing x. It follows that x is nilpotent and so vanishes on R_{red}. Since $[I, x] \in S$ is the image of $x \in R$, we have that $R_{red} \to S$ is an isomorphism.

As X is locally of finite type, [Stacks, Tag 01ZC] gives $X_{dR}(R) \simeq X(S) \simeq X(R_{red})$, concluding the proof. □

Perhaps not surprisingly, given the aforementioned geometric interpretation of X_{dR}, formal completions of schemes can be written in terms of de Rham spaces.

Proposition A.4. *Let X be a k-scheme and let Z be a closed subscheme of X. The formal completion $\widehat{X}_Z$ of X along Z is isomorphic to $X \times_{X_{dR}} Z_{dR}$.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defined by Z, and let R be a k-algebra. We aim to obtain a functorial isomorphism

$$\operatorname*{colim}_{I \subset R} X(R) \times_{X(R/I)} Z(R/I) \simeq \operatorname*{colim}_{n \geqslant 0} \operatorname{Spec}_X(\mathcal{O}_X/\mathcal{I}^{n+1})(R),$$

where the colimit on the left runs through the nilpotent ideals of R. For an ideal $I \subset R$, let i_I denote the closed immersion $\operatorname{Spec} R/I \to \operatorname{Spec} R$. Then, we have that

$$\begin{aligned}
\operatorname*{colim}_{I \subset R} X(R) \times_{X(R/I)} Z(R/I) &\simeq \operatorname*{colim}_{I \subset R} \{x \in X(R) \mid i_I^* x^* \mathcal{I} = 0\} \\
&\simeq \operatorname*{colim}_{n \geqslant 0} \operatorname*{colim}_{I^{n+1}=0} \{x \in X(R) \mid i_I^* x^* \mathcal{I} = 0\} \\
&\simeq \operatorname*{colim}_{n \geqslant 0} \{x \in X(R) \mid x^* \mathcal{I}^{n+1} = 0\} \simeq \widehat{X}_Z(R),
\end{aligned}$$

where the last isomorphism is the universal property of the relative spectrum. □

Note that the expression $X \times_{X_{dR}} Z_{dR}$ is meaningful even if $Z \to X$ is not a closed immersion. In such cases, it can be used to *define* formal completions. Moreover, this characterization makes it clear that the projection $\widehat{X}_Z \to X$ is a monomorphism whenever $Z \to X$ is.

Corollary A.5. *If Z is a closed subscheme of a k-scheme X, the projection $\widehat{X}_Z \to X$ is a monomorphism of presheaves.*

Proof. According to [Stacks, Tag 01L7], the closed immersion $i \colon Z \to X$ is a monomorphism in the category of schemes. Now, the Yoneda embedding preserves limits and any functor that preserves limits preserves monomorphisms [Stacks, Tag 01L3]. In other words, i is a monomorphism of presheaves. Since the de Rham functor preserves finite limits, so is $Z_{dR} \to X_{dR}$. Finally, fibered products preserve monomorphisms and this finishes the proof. □

For a morphism of k-schemes $p \colon X \to S$, the universal property of fibered products induces a map $\iota_p \colon X \to X_{dR} \times_{S_{dR}} S$. As the following proposition shows, this map faithfully encodes the differential information contained in p.[11]

Proposition A.6. *Let $p \colon X \to S$ be a morphism of k-schemes. Then p is formally smooth (resp. formally unramified) if and only if $\iota_p \colon X \to X_{dR} \times_{S_{dR}} S$ is an epimorphism (resp. monomorphism) of presheaves.*

[11] This result is somewhat akin to the slogan "f is smooth (resp. unramified) if and only if df is surjective (resp. injective)".

Proof. Recall that p is formally smooth (resp. formally unramified) if, for every k-algebra R with a map $\operatorname{Spec} R \to S$ and for every nilpotent ideal $I \subset R$, the induced map

$$\operatorname{Hom}_S(\operatorname{Spec} R, X) \to \operatorname{Hom}_S(\operatorname{Spec} R/I, X)$$

is surjective (resp. injective). We will translate this condition into the surjectivity (resp. injectivity) of $X(R) \to (X_{dR} \times_{S_{dR}} S)(R)$.

Suppose that p is formally smooth, let R be any k-algebra, and let $[x, s]$ be an element of

$$(X_{dR} \times_{S_{dR}} S)(R) \simeq \operatorname{colim}_{I \subset R} X(R/I) \times_{S(R/I)} S(R).$$

That is, there exists a nilpotent ideal $I \subset R$ such that $x \in X(R/I)$ and $s \in S(R)$ agree on $S(R/I)$. By formal smoothness, we have a morphism $\overline{x}\colon \operatorname{Spec} R \to X$ making the diagram

$$\begin{array}{ccc} \operatorname{Spec} R/I & \xrightarrow{x} & X \\ \downarrow & \overset{\overline{x}}{\nearrow} & \downarrow p \\ \operatorname{Spec} R & \xrightarrow{s} & S \end{array}$$

commute. (The lower triangle commutes since $\overline{x}$ is a morphism over S, and the upper triangle commutes because $\overline{x}$ maps to x.) This is an element of $X(R)$ mapping to $(X_{dR} \times_{S_{dR}} S)(R)$. Conversely, suppose that R is a k-algebra with a map $s\colon \operatorname{Spec} R \to S$, $I \subset R$ is a nilpotent ideal, and x is an element of $\operatorname{Hom}_S(\operatorname{Spec} R/I, X)$. This data defines an element of $(X_{dR} \times_{S_{dR}} S)(R)$ and so there exists $\overline{x} \in X(R)$ mapping to it. In other words, $\iota_p\colon X \to X_{dR} \times_{S_{dR}} S$ is an epimorphism if and only if p is formally smooth.

Now, if p is formally unramified, consider a k-algebra R and let $x, y \in X(R)$ be two elements whose images in $X_{dR} \times_{S_{dR}} S(R)$ coincide. That is, there exists a nilpotent ideal $I \subset R$ such that the diagram

$$\operatorname{Spec} R/I \to \operatorname{Spec} R \rightrightarrows X \xrightarrow{p} S$$

commutes. In particular, x and y define elements of $\operatorname{Hom}_S(\operatorname{Spec} R, X)$ that coincide on $\operatorname{Hom}_S(\operatorname{Spec} R/I, X)$. Since p is formally unramified, we have that $x = y$.

Suppose that $\iota_p\colon X \to X_{dR} \times_{S_{dR}} S$ is a monomorphism and let $x, y \in \operatorname{Hom}_S(\operatorname{Spec} R, X)$ be two morphisms that coincide on $\operatorname{Hom}_S(\operatorname{Spec} R/I, X)$, for some k-algebra R with a map $\operatorname{Spec} R \to S$ and a nilpotent ideal $I \subset R$. In particular, x and y are elements of $X(R)$ that coincide on $(X_{dR} \times_{S_{dR}} S)(R)$. It follows that $x = y$ and so p is formally unramified. □

Let $Y \to X$ be an immersion of k-schemes that factors as $Y \to U \to X$, where $Y \to U$ is a closed immersion with ideal $\mathscr{I}$ and $U \to X$ is an open immersion. The formal completion of X along Y is usually defined as the colimit of $\operatorname{Spec}_U(\mathcal{O}_U/\mathcal{I}^{n+1})$, for $n \geqslant 0$. The previous proposition implies that $U \simeq U_{dR} \times_{X_{dR}} X$, and so

$$U \times_{U_{dR}} Z_{dR} \simeq X \times_{X_{dR}} U_{dR} \times_{U_{dR}} Z_{dR} \simeq X \times_{X_{dR}} Z_{dR},$$

proving that Proposition A.4 also applies to locally closed immersions.

Corollary A.7. *Let* $X \to S$ *be a formally smooth morphism of* k*-schemes. Then* $X_{dR} \times_{S_{dR}} S$ *is the coequalizer of*

$$\widehat{(X \times_S X)}_\Delta \rightrightarrows X,$$

where $\widehat{(X \times_S X)}_\Delta$ *is the formal completion of* $X \times_S X$ *along the diagonal.*

Proof. In order to simplify notation, let $Y = X_{dR} \times_{S_{dR}} S$. Since every epimorphism is effective in a topos, $X \to Y$ is the coequalizer of $X \times_Y X \rightrightarrows X$. Now, the result follows from general category theory: the pullback of

$$\begin{array}{ccc} & & X \times_S X \\ & & \downarrow \\ X_{dR} & \xrightarrow{\Delta_{dR}} & (X \times_S X)_{dR} \end{array}$$

is $X \times_Y X$. □

Recall that the functor of points of a scheme X is a sheaf for the étale and fppf topologies. We will now study the descent properties of X_{dR}, and we begin with a lemma.

Lemma A.8. *Let* R *be a* k*-algebra and let* $\{R \to R_i\}_{i \in I}$ *be an étale covering. Then the reduction* $\{R_{red} \to R_{i,red}\}_{i \in I}$ *is also an étale covering. Moreover, any étale covering of* R_{red} *arises in this way.*

Proof. Since $R \to R_i$ is étale, so is its base change $R_{red} \to R_{red} \otimes_R R_i$. By [Stacks, Tag 033B], we have that $R_{red} \otimes_R R_i$ is reduced and then [EGA I, Cor. 5.1.8] gives that

$$R_{red} \otimes_R R_i = (R_{red} \otimes_R R_i)_{red} \simeq (R_{red} \otimes_{R_{red}} R_{i,red})_{red} \simeq R_{i,red}.$$

It follows that $\{R_{red} \to R_{i,red}\}_{i \in I}$ is an étale covering of R_{red}.

Now, consider an étale covering $\{R_{red} \to S_i\}_{i \in I}$ of R_{red}. By the topological invariance of the étale site, there exists a covering $\{R \to R_i\}_{i \in I}$ along with isomorphisms $R_{red} \otimes_R R_i \simeq S_i$ for all $i \in I$ [Stacks, Tag 04DZ]. The same argument as above shows that S_i is reduced, and then $S_i \simeq R_{i,red}$. □

Proposition A.9. *Let* X *be a locally of finite type scheme over* k*. The de Rham space* X_{dR} *is an étale sheaf on* Aff/k*.*

Proof. Let R be a k-algebra and let $\{R \to R_i\}_{i \in I}$ be an étale covering of R. We want to prove that the diagram

$$X(R_{red}) \to \prod_i X(R_{i,red}) \rightrightarrows \prod_{i,j} X((R_i \otimes_R R_j)_{red})$$

is an equalizer. The lemma above says that $\{R_{\mathrm{red}} \to R_{i,\mathrm{red}}\}_{i\in I}$ is also an étale cover and then the fact that X is an étale sheaf implies that the diagram

$$X(R_{\mathrm{red}}) \to \prod_i X(R_{i,\mathrm{red}}) \rightrightarrows \prod_{i,j} X(R_{i,\mathrm{red}} \otimes_{R_{\mathrm{red}}} R_{j,\mathrm{red}})$$

is an equalizer. The same argument as in the proof of the previous lemma shows that $R_{i,\mathrm{red}} \otimes_{R_{\mathrm{red}}} R_{j,\mathrm{red}}$ is reduced. Then [EGA I, Cor. 5.1.8] gives isomorphisms $R_{i,\mathrm{red}} \otimes_{R_{\mathrm{red}}} R_{j,\mathrm{red}} \simeq (R_i \otimes_R R_j)_{\mathrm{red}}$, finishing the proof. $\square$

In particular, the preceding proposition implies that the de Rham space of a commutative algebraic group over k is an abelian étale sheaf on Aff/k.

Proposition A.10. *Suppose that k has characteristic zero. Then the functor $(-)_{\mathrm{dR}}$ from commutative algebraic groups over k to abelian étale sheaves on Aff/k is exact.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence of commutative algebraic groups over k. In particular, it is left-exact in the category of abelian *presheaves* on Aff/k. By Proposition A.2, the induced exact sequence $0 \to A_{\mathrm{dR}} \to B_{\mathrm{dR}} \to C_{\mathrm{dR}} \to 0$ is also left-exact in abelian presheaves. Since sheafification is exact, this sequence is left-exact in the category of abelian étale sheaves.

Let us verify that $B_{\mathrm{dR}} \to C_{\mathrm{dR}}$ is an epimorphism of abelian sheaves. Given a k-algebra R and an element $c \in C_{\mathrm{dR}}(R) = C(R_{\mathrm{red}})$, the fact that $B \to C$ is an epimorphism of étale sheaves implies that there exists a covering $\{R_{\mathrm{red}} \to S_i\}_{i\in I}$ such that $c|_{S_i}$ is in the image of $B(S_i) \to C(S_i)$ for all $i \in I$ [Stacks, Tag 00WN]. Lemma A.8 then gives a covering $\{R \to R_i\}_{i\in I}$ whose reduction is $\{R_{\mathrm{red}} \to S_i\}_{i\in I}$. It follows that $c|_{R_i} = c|_{S_i}$ is in the image of $B_{\mathrm{dR}}(R_i) \to C_{\mathrm{dR}}(R_i)$ for all $i \in I$, concluding the proof. $\square$

We proved in Corollary A.7 that X_{dR} is a quotient of X in which we identify infinitesimally close points. When X is a commutative algebraic group G, the difference of two such points has to live in an infinitesimal neighborhood of the identity. This heuristic leads to the result below.

Proposition A.11. *Let G be a smooth commutative algebraic group over k. Then G_{dR} is isomorphic to the presheaf quotient $G/\widehat{G}$, where $\widehat{G}$ is the formal completion of G along the identity. In particular, G_{dR} is also isomorphic to the sheaf quotient $G/\widehat{G}$.*

Proof. In this proof, let us consider every (co)limit to be taken inside the category of abelian presheaves on Aff/k. As the cokernel of the identity section $e\colon \operatorname{Spec} k \to G$ is G itself, a variant of Proposition A.2 for abelian presheaves shows that the cokernel of $e_{\mathrm{dR}}\colon \operatorname{Spec} k \to G_{\mathrm{dR}}$ is G_{dR}. The universal property of cokernels then induces the dashed map below.

$$\begin{array}{ccccc} \widehat{G} & \longrightarrow & G & \longrightarrow & G/\widehat{G} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{e_{dR}} & G_{dR} & = & G_{dR} \end{array}$$

The square on the left is cartesian due to Proposition A.4, and $G \to G_{dR}$ is an epimorphism since G is smooth. Then, [Stacks, Tag 08N4] implies that the square on the left is also cocartesian, and [Stacks, Tag 08N3] gives that $G/\widehat{G} \to G_{dR}$ is an isomorphism. Since G_{dR} is already an étale sheaf, the presheaf and the sheaf quotients $G/\widehat{G}$ coincide. □

We now study the descent properties of de Rham spaces with respect to the finer fppf topology. We remark that Proposition A.10 as well as the following proposition and its corollary, are the unique results in this section that need the base field k to have characteristic zero.

PROPOSITION A.12. *Let G be a commutative algebraic group over a characteristic zero field k. Then G_{dR} is an fppf sheaf isomorphic to $G/\widehat{G}$ and the functor $(-)_{dR}$ from commutative algebraic groups over k to abelian fppf sheaves is exact.*

Proof. According to Proposition 2.11, the formal completion $\widehat{G}$ is a direct sum of copies of $\widehat{\mathbb{G}}_a$. Then, given a k-algebra R, [Bha22, Rem. 2.2.18] says that $H^1_{fppf}(R, \widehat{\mathbb{G}}_a) = 0$ and so $(G/\widehat{G})(R) \simeq G(R)/\widehat{G}(R) \simeq G_{dR}(R)$, where the quotient on the left is taken on the fppf topology. In other words G_{dR} is an fppf sheaf isomorphic to $G/\widehat{G}$. The exactness of $(-)_{dR}$ here is a particular case of Proposition A.10. □

COROLLARY A.13. *Let X be a locally of finite type scheme over a characteristic zero field k. Then X_{dR} is an fppf sheaf.*

Proof. Let R be a k-algebra and let $\{R \to R_i\}_{i \in I}$ be an fppf covering of R. By the previous proposition, the diagram

$$R_{red} \to \prod_i R_{i,red} \rightrightarrows \prod_{i,j} (R_i \otimes_R R_j)_{red}$$

is an equalizer in the category of k-algebras. The functor of points $X(-)\colon (\mathsf{Aff}/k)^{op} \to \mathsf{Set}$ sends this diagram to an equalizer in the category of sets, finishing the proof. □

■ REMARK A.14 — DE RHAM SPACES IN POSITIVE CHARACTERISTIC. Let k be a field of characteristic $p > 0$. Given a k-algebra R, the colimit R_{perf} of the tower

$$R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} \cdots$$

is the so-called *colimit perfection* of R. It is always a perfect k-algebra and the natural map $R \to R_{perf}$ is universal among morphisms from R to a perfect algebra. We define a presheaf $\mathbb{G}_{a,\mathrm{perf}}$ on Aff/k by $\mathbb{G}_{a,\mathrm{perf}}(R) := R_{\mathrm{perf}}$. As [Bha22, Rem. 2.2.18] shows, we have an exact sequence of abelian fppf sheaves

$$0 \to \widehat{\mathbb{G}}_a \to \mathbb{G}_a \to \mathbb{G}_{a,\mathrm{perf}} \to 0.$$

It follows that the natural map of abelian étale sheaves $\mathbb{G}_{a,\mathrm{dR}} \to \mathbb{G}_{a,\mathrm{perf}}$ identifies $\mathbb{G}_{a,\mathrm{perf}}$ with the fppf sheafification of $\mathbb{G}_{a,\mathrm{dR}}$. ∎

We end this section by extending the definition of de Rham spaces from functors on Aff/k to functors on Sch/k.

Definition A.15. Let X be a scheme over k, seen as its functor of points $(\mathsf{Sch}/k)^{\mathrm{op}} \to \mathsf{Set}$. We define its *de Rham space* by taking the de Rham space of the restriction $(\mathsf{Aff}/k)^{\mathrm{op}} \to (\mathsf{Sch}/k)^{\mathrm{op}} \to \mathsf{Set}$ and then right Kan extending to $(\mathsf{Sch}/k)^{\mathrm{op}}$.

This definition actually coincides with the naive one when X is locally of finite type over k, but it will be more convenient. Indeed, given a k-scheme S, Proposition A.3 implies that

$$X_{\mathrm{dR}}(S) \simeq \lim_{\mathrm{Spec}\, R \to S} X(\mathrm{Spec}\, R_{\mathrm{red}}),$$

where the limit runs through the affine k-schemes with a map to S. Since the functor $X(-) = \mathrm{Mor}_k(-, X)$ commutes with limits, this is also

$$X\left(\operatorname*{colim}_{\mathrm{Spec}\, R \to S} \mathrm{Spec}\, R_{\mathrm{red}}\right) \simeq X(S_{\mathrm{red}}).$$

The usefulness of this definition comes from the fact that we have an equivalence of topoi $\mathsf{Sh}((\mathsf{Aff}/k)_{\text{ét}}) \simeq \mathsf{Sh}((\mathsf{Sch}/k)_{\text{ét}})$. Here, the functor $\mathsf{Sh}((\mathsf{Sch}/k)_{\text{ét}}) \to \mathsf{Sh}((\mathsf{Aff}/k)_{\text{ét}})$ is given by restriction and the functor $\mathsf{Sh}((\mathsf{Aff}/k)_{\text{ét}}) \to \mathsf{Sh}((\mathsf{Sch}/k)_{\text{ét}})$ is a right Kan extension [Stacks, Tag 021E].

Every result in this section that was true for sheaves generalizes to this setting. Take Proposition A.11 as an example: it is no longer true that $G_{\mathrm{dR}} \simeq G/\widehat{G}$ as presheaves on Sch/k, but G_{dR} is isomorphic to the quotient $G/\widehat{G}$ taken in $\mathsf{Ab}((\mathsf{Sch}/k)_{\text{ét}})$. For the sake of completeness, we give precise statements below.

Proposition A.16. *Let Z be a closed subscheme of a k-scheme X. Then the formal completion $\widehat{X}_Z$ of X along Z is isomorphic to $X \times_{X_{\mathrm{dR}}} Z_{\mathrm{dR}}$ and the projection $\widehat{X}_Z \to X$ is a monomorphism of étale sheaves on* Sch/k.

Proposition A.17. *Let $f\colon X \to S$ be a morphism of k-schemes. Then f is formally unramified if and only if $X \to X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$ is a monomorphism of étale sheaves on* Sch/k. *Moreover, if f is formally smooth, then $X \to X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$ is an epimorphism of étale sheaves on* Sch/k.

Proposition A.18. *Let* G *be a commutative algebraic group over* k*. Then* G_{dR} *is isomorphic to the quotient* $G/\widehat{G}$ *taken in* $\mathsf{Ab}((\mathsf{Sch}/k)_{\text{ét}})$*. Moreover, the functor* $(-)_{dR}$ *from commutative algebraic groups over* k *to abelian étale sheaves on* Sch/k *is exact.*

Due to [Stacks, Tag 021V], all results in the last 3 propositions also hold for the fppf topology as long as k has characteristic zero.

A.2. Connections as torsors on the de Rham space

Let k be a field of characteristic zero, S be a k-scheme, and X be a smooth S-scheme. For a vector bundle V on X, recall that a connection on V relative to S is an $\mathcal{O}_S$-linear map $\nabla\colon V \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} V$ satisfying the Leibniz rule

$$\nabla(fx) = df \otimes x + f\nabla(x),$$

for local sections f of $\mathcal{O}_X$ and x of V. The connection ∇ is said to be *flat* if $\nabla_1 \circ \nabla = 0$, where $\nabla_1\colon \Omega^1_{X/S} \otimes_{\mathcal{O}_X} V \to \Omega^2_{X/S} \otimes_{\mathcal{O}_X} V$ is defined by

$$\nabla_1(\omega \otimes x) = d\omega \otimes x - \omega \wedge \nabla(x).$$

The following result allows us to study those objects as torsors on de Rham spaces.

Proposition A.19. *Let* $p\colon X \to S$ *be a smooth morphism of schemes, and let* $\iota_p\colon X \to X_{dR} \times_{S_{dR}} S$ *be the induced morphism as in Proposition A.6. For an integer* n*, there exists an equivalence of groupoids (indicated by the dashed arrow below) making the diagram*

$$\begin{array}{ccccc}
\left\{\begin{matrix}\mathrm{GL}_n\text{-torsors} \\ \text{over } X_{dR} \times_{S_{dR}} S\end{matrix}\right\} & \overset{\sim}{\dashrightarrow} & \left\{\begin{matrix}\text{Rank } n \text{ vector bundles on } X \\ \text{with flat connection relative to } S\end{matrix}\right\} & & (V,\nabla) \\
{\scriptstyle \iota_p^*}\downarrow & & \downarrow & & \downarrow \\
\left\{\begin{matrix}\mathrm{GL}_n\text{-torsors} \\ \text{over } X\end{matrix}\right\} & \xrightarrow{\ \sim\ } & \left\{\begin{matrix}\text{Rank } n \text{ vector} \\ \text{bundles on } X\end{matrix}\right\} & & V
\end{array}$$

commute up to isomorphism. Moreover, this equivalence is functorial in p*. For* $n = 1$*, the equivalence is symmetric monoidal with respect to the contracted product of torsors and the tensor product of connections.*

Proof. Equip the category $\mathsf{Sh}((\mathsf{Sch}/S)_{\mathrm{fppf}})$ with its canonical topology, in which coverings are given by jointly epimorphic families. By Proposition A.17, the morphism ι_p is an epimorphism, and since torsors form a stack on $\mathsf{Sh}((\mathsf{Sch}/S)_{\mathrm{fppf}})$, the groupoid of GL_n-torsors over $X_{dR} \times_{S_{dR}} S$ is equivalent to the groupoid of descent data $\mathrm{Desc}(\iota_p)$.

An object of $\mathrm{Desc}(\iota_p)$ consists of a GL_n-torsor V over X together with an isomorphism

$$\varepsilon\colon\ \mathrm{pr}_1^* V \to \mathrm{pr}_2^* V,$$

satisfying the cocycle condition $\mathrm{pr}_{12}^*(\varepsilon) \circ \mathrm{pr}_{23}^*(\varepsilon) = \mathrm{pr}_{13}^*(\varepsilon)$. Here we set $Y := X_{dR} \times_{S_{dR}} S$, and we write pr_i (resp. pr_{ij}) for the canonical projection to the i-th factor (resp. to the (i,j)-factors) in the simplicial diagram

$$X \times_Y X \times_Y X \substack{\rightarrow\\\rightarrow\\\rightarrow} X \times_Y X \rightrightarrows X.$$

As in the proof of Corollary A.7, the projection maps appearing above can be identified with the canonical projections

$$(\widehat{X \times_S X \times_S X})_\Delta \substack{\rightarrow\\\rightarrow\\\rightarrow} (\widehat{X \times_S X})_\Delta \rightrightarrows X,$$

where $(\widehat{-})_\Delta$ denotes the formal completion along the diagonal.

A morphism $\varphi\colon (V, \varepsilon) \to (V', \varepsilon')$ in $\mathrm{Desc}(\iota_p)$ consists of a morphism of GL_n-torsors $\varphi\colon V \to V'$ making the diagram

$$\begin{array}{ccc} \mathrm{pr}_1^* V & \xrightarrow{\mathrm{pr}_1^*(\varphi)} & \mathrm{pr}_1^* V' \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon'} \\ \mathrm{pr}_2^* V & \xrightarrow{\mathrm{pr}_2^*(\varphi)} & \mathrm{pr}_2^* V' \end{array}$$

commute. Using the equivalence between GL_n-torsors and rank n vector bundles, we conclude that $\mathrm{Desc}(\iota_p)$ is equivalent to the groupoid of rank n vector bundles on X endowed with a stratification relative to S in the sense of [BO78, Def. 2.10]. The result then follows from [BO78, Thm. 2.15]. □

In this paper, the preceding result will be used primarily through the following immediate corollary. For the reader's convenience, we note that $X_{dR} \times S \simeq (X \times S)_{dR} \times_{S_{dR}} S$.

Corollary A.20. *Let X be a smooth scheme over a field k of characteristic zero, and let S be a k-scheme. Then the fppf cohomology group*

$$H^1(X_{dR} \times S, \mathbb{G}_m)$$

classifies isomorphism classes of line bundles on X endowed with an integrable connection relative to S. The group law corresponds to the tensor product of connections.

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