Homological Algebra

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Introduction

Gabriel-from-the-future here. I originally wrote these notes for myself when I was first learning homological algebra. They don't go very far, but I still haven't quite found an introductory treatment that follows the same guiding principles.

In particular, I discussed abelian categories on their own terms, without pretending that they are categories of modules over a ring. I took derived categories as the starting point, rather than presenting them only after the classical approach. And I have included detailed proofs throughout.

The notes stop a bit before derived functors, but I still think there are a few ideas here that may be useful. Back in the day, some friends seemed to appreciate these notes, so I'm making them publicly available in the hope that they might help someone else to learn this beautiful subject.

Contents

1 Abelian categories

Homological algebra deals extensively with the notions of kernel, image, exact sequences, chain complexes, and the like. This chapter will explain the most general setting, that of abelian categories, in which these concepts make sense. Certainly, the category of A-modules has all the needed characteristics. Going even further, it is true that every abelian category has a fully faithful embedding on A-Mod for some (not necessarily commutative) ring A. However, when it is not too troublesome, we'll study abelian categories "on their own" for we believe that understanding arrow-theoretic arguments and not becoming dependent on a difficult theorem can only be beneficial.

1.1 Additive categories

We begin our quest of understanding which properties a category should have in order for exact sequences to make sense. A first problem is that our category should have a distinguished object corresponding to the trivial module in A-Mod. In order to allow for exact sequences, this object should be initial and final at the same time. We arrive at our first definition.

Definition 1.1.1 Let A be a category. A *zero-object* is an object of A which is both initial and final. We'll always denote zero-objects as 0.

The reader should notice that even reasonable categories may fail to have initial or final objects (the category of fields, for example, has neither). And even if they exist, they may not coincide (as in Set or Ring). Nevertheless, Grp, Ab, and A-Mod are examples of categories possessing zero-objects.

The existence of zero-objects in a category allows us to talk about zero-morphisms.

Definition 1.1.2 Let A be a category with a zero-object 0. A morphism $\varphi : M \to N$ is called a *zero-morphism* if it factors through the zero-object 0. We'll also denote zero-morphisms by 0.

We observe that in a category with a zero-object, there is exactly one zero-morphism from each object M to each object N: it's just the composite of the unique morphism $M \to 0$ with the unique morphism $0 \to N$. In any of the aforementioned categories which possess zero-objects, the zero morphism $M \to N$ is the one sending every element of M to $0 \in N$. Moreover, the composition of a zero-morphism with an arbitrary morphism is again a zero-morphism. Indeed, the composition factors through 0.

In an abstract category, we have no means of defining kernels set-theoretically as the subobjects composed of the elements which are sent to zero. Instead, we define a kernel as a *morphism* by a suitable universal property.

Definition 1.1.3 — Kernel. Let $\varphi : M \to N$ be a morphism in a category A with a zero-object 0. The *kernel* of φ is the equalizer of φ and the zero-morphism. In other words, it is a morphism $\iota : K \to M$ such that, whenever $\zeta : Z \to M$ satisfies $\varphi \circ \zeta = 0$, there exists a unique morphism $Z \to K$ making the diagram

$$\begin{tikzcd}
K \arrow[r, "\iota"] & M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \\
Z \arrow[u, dashed] \arrow[ru, "\zeta"'] & &
\end{tikzcd}$$

commute. We denote both K and $\iota : K \to M$ by $\ker \varphi$.

Once again, we observe that kernels are not guaranteed to exist even in reasonable categories. For example, kernels may fail to exist in the category of finitely generated A-modules whenever A is not noetherian.

In any of the previously mentioned categories with zero-objects, the universal property of the kernel is satisfied by the inclusion map from the set-theoretic kernel. This generalizes nicely to the categorical kernel. For that, we need another piece of nomenclature.

Definition 1.1.4 — Subobject. Let M be an object in a category A. We say that two monomorphisms $s : S \to M$ and $t : T \to M$ are equivalent if there exists an isomorphism $S \to T$ making the diagram

$$\begin{tikzcd}
S \arrow[rr, "\sim"] \arrow[dr, hook, "s"'] & & T \arrow[dl, hook', "t"] \\
& M &
\end{tikzcd}$$

commute. In other words, s and t are equivalent if they are isomorphic in the slice category $A \downarrow M$. A *subobject* of M is an equivalence class for this equivalence relation.

The universal property of kernels implies that all kernels of a morphism $M \to N$ belong to the same isomorphism class in $A \downarrow M$. Thus, in order to prove that the kernel of $M \to N$ is a subobject of M it suffices to show that kernels are monic.

Proposition 1.1.1 Let $\varphi : M \to N$ be a morphism in a category A with a zero-object 0 and suppose that $\ker \varphi : K \to M$ is its kernel. Then $\ker \varphi$ is a monomorphism and so defines a subobject of M.

The reader should notice that the same proof shows that every equalizer is monic.

Proof. Let $\alpha, \beta : Z \to K$ be two morphisms such that $(\ker \varphi) \circ \alpha = (\ker \varphi) \circ \beta$ and let ζ be their common compositions. By the universal property of kernels, there is a unique morphism $Z \to K$ making the diagram

$$\begin{tikzcd}
K \arrow[r, "\ker \varphi"] & M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \\
Z \arrow[u] \arrow[ur, "\zeta"'] & &
\end{tikzcd}$$

commute. But α and β are two such morphisms. It follows that $\alpha = \beta$. □

In most categories in algebra, kernels measure how far a morphism is from being injective. The following propositions shows that the categorical kernel still, in some sense, encodes this information.

Proposition 1.1.2 Let $\varphi : M \to N$ be a monomorphism in a category A with a zero-object 0. Then $\ker \varphi$ is the zero-morphism $0 \to M$.

Proof. Suppose $\zeta : Z \to M$ is a morphism such that $\varphi \circ \zeta = 0$. Since φ is a monomorphism, $\varphi \circ \zeta = 0 = \varphi \circ 0$ means that $\zeta = 0$ and so ζ factors uniquely through the zero-object, making the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \\
Z \arrow[u, dashed] \arrow[ur, "\zeta"'] & &
\end{tikzcd}$$

commute. This means that $0 \to M$ is the, necessarily unique, kernel of φ. □

Proposition 1.1.3 Let $\varphi : M \to N$ be a morphism in a category A with a zero-object 0. Then φ is a zero-morphism if and only if $\ker \varphi$ is, up to isomorphism, the identity on M.

Proof. Suppose φ is the zero-morphism. Then $\varphi \circ \mathrm{id}_M = 0 \circ \mathrm{id}_M$ and so any morphism $\zeta : Z \to M$ factors uniquely through id_M. Conversely, if id_M is a kernel of φ, then $\varphi = \varphi \circ \mathrm{id}_M = 0$. □

The main problems of the categorical kernel are the fact that they may not exist and, even when they exist, it is not necessarily true that every monomorphism is a kernel, as in A-Mod. For example, in the category of groups, kernels are normal subgroups but monomorphisms correspond to all subgroups. All these problems will be solved in the next section. For now, we observe that the dual notion (which inverses all the arrows) of kernel is just as useful.

Definition 1.1.5 — Cokernel. Let $\varphi : M \to N$ be a morphism in a category A with a zero-object 0. The *cokernel* of φ is the coequalizer of φ and the zero-morphism. In other words, it is a morphism $\pi : N \to C$ such that, whenever $\beta : N \to Z$ satisfies $\beta \circ \varphi = 0$, there exists a unique morphism $C \to Z$ making the diagram

$$\begin{tikzcd}
& & Z \\
M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \arrow[ur, "\beta"] \arrow[r, "\pi"] & C \arrow[u, dashed]
\end{tikzcd}$$

commute. We denote both C and $\pi : N \to C$ by $\operatorname{coker} \varphi$.

Before we prove any properties of the cokernel, we present how it works in some categories, since the reader may be unfamiliar with it.

■ **Example 1.1.1 — Cokernels in A-Mod.** Let $\varphi : M \to N$ be a morphism of A-modules. Here, the cokernel of φ is the quotient map $\pi : N \to N/\operatorname{im}\varphi$, where $\operatorname{im}\varphi$ is the usual set-theoretic image. Indeed, if $\beta : N \to P$ satisfies $\beta \circ \varphi = 0$, then $\operatorname{im}\varphi \subset \ker \beta$ and the universal property of the quotient induces a unique morphism $\widetilde{\beta} : N/\operatorname{im}\varphi \to P$ which makes the diagram

$$\begin{tikzcd}
& & P \\
M \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "0"'] & N \arrow[ur, "\beta"] \arrow[r, "\pi"] & N/\operatorname{im}\varphi \arrow[u, dashed, "\widetilde{\beta}"']
\end{tikzcd}$$

commute. In other words, $\pi : N \to N/\operatorname{im}\varphi$ satisfies the universal property of the cokernel. ■

■ **Example 1.1.2 — Cokernels in Grp.** Let $\varphi : G \to H$ be a morphism of groups. The same argument as in A-Mod doesn't work as the set-theoretical image may not be a normal subgroup of H. Nevertheless, we may consider the smallest normal subgroup of H containing $\operatorname{im}\varphi$, which we denote by N. Then the cokernel of φ becomes the quotient map $\pi : H \to H/N$. Indeed, if $\beta : H \to H'$ satisfies $\beta \circ \varphi = 0$, then $\operatorname{im}\varphi \subset \ker\beta$ and, since $\ker\beta$ is a normal subgroup of H containing $\operatorname{im}\varphi$, $N \subset \ker\beta$. Now the same argument as before works, showing that $\pi : H \to H/N$ satisfies the universal property of the cokernel. ■

■ **Example 1.1.3 — Cokernels in the category of Banach spaces.** The same problem as before happens frequently in topological settings. In the category of Banach spaces with bounded (continuous) linear maps as morphisms, not every subspace defines a quotient, only the closed ones. A similar reasoning as before shows that the cokernel of a morphism $T : X \to Y$ is the quotient map $Y \to Y/N$, where N is the closure of the set-theoretical image $\operatorname{im} T$ in Y. ■

It is actually the case that, whenever it exists, the cokernel of a morphism $\varphi : M \to N$ is a quotient of N just as the kernel is a subobject of M. In order to make sense of that in an arbitrary category, we invert the arrows in the definition 1.1.4.

Definition 1.1.6 — Quotient object. Let N be an object in a category A. We say that two epimorphisms $p : N \to S$ and $q : N \to T$ are equivalent if there exists an isomorphism $S \to T$ making the diagram

$$\begin{tikzcd} & N \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S \arrow[rr, "\sim"] & & T \end{tikzcd}$$

commute. In other words, p and q are equivalent if they are isomorphic in the coslice category $N \downarrow A$. A *quotient object* of N is an equivalence class for this equivalence relation.

As before, it is clear by the universal property that all cokernels of a morphism $M \to N$ belong to the same isomorphism class in $N \downarrow A$. So, by proving that cokernels are epic, we prove that every cokernel is a quotient object.

Proposition 1.1.4 Let $\varphi : M \to N$ a morphism in a category A with a zero-object 0 and suppose that $\pi : N \to C$ is its cokernel. Then π is an epimorphism and so $\operatorname{coker} \varphi$ is a quotient object of N.

Proof. We could do basically the same argument as in the proof of proposition 1.1.1, but we'll use this as an opportunity to understand a powerful idea: the duality principle. Let $\alpha, \beta : C \to D$ be morphisms such that $\alpha \circ \pi = \beta \circ \pi$. Inverting all the arrows, we see that $\pi^{op} : C \to N$ is the kernel of $\varphi^{op} : N \to M$ and $\pi^{op} \circ \alpha^{op} = \pi^{op} \circ \beta^{op}$. Since π^{op} is a monomorphism by proposition 1.1.1, $\alpha^{op} = \beta^{op}$ and so $\alpha = \beta$, proving that π is an epimorphism. ☐

By inverting all the arrows as above, we can easily prove dual versions of the propositions 1.1.2 and 1.1.3, which we state below.

Proposition 1.1.5 Let $\varphi : M \to N$ be an epimorphism in a category A with a zero-object 0. Then $\operatorname{coker} \varphi$ is the zero morphism $N \to 0$.

Proposition 1.1.6 Let $\varphi : M \to N$ be a morphism in a category A with a zero-object. Then φ is a zero-morphism if and only if $\operatorname{coker} \varphi$ is, up to isomorphism, the identity on N.

Everything we did so far only makes sense given the existence of zero-morphisms in the category under consideration. There's a natural way in which a category may be endowed with such morphisms.

> **Definition 1.1.7 — Preadditive category.** A category A is said to be *preadditive* if each set of morphisms $\mathrm{Hom}_{\mathsf{A}}(M, N)$ is endowed with an abelian group structure, in such a way that the composition maps are bilinear.

The exquisite reader may recognize that this is nothing but a category enriched over Ab. Explicitly, in a preadditive category it makes sense to add or subtract morphisms and this operation satisfies

$$\varphi \circ (\psi_1 + \psi_2) = \varphi \circ \psi_1 + \varphi \circ \psi_2 \qquad \text{and} \qquad (\varphi_1 + \varphi_2) \circ \psi = \varphi_1 \circ \psi + \varphi_2 \circ \psi,$$

whenever those compositions exist.

A preadditive category A may still lack zero-objects. But, given a zero-object, we have two natural notions of zero-morphism $M \to N$: the unique morphism $M \to N$ which factors through the zero object and the identity of $\mathrm{Hom}_{\mathsf{A}}(M, N)$. It is reassuring to know that they coincide.

> **Proposition 1.1.7** In a preadditive category A, the following conditions are equivalent:
>
> (a) A has an initial object;
>
> (b) A has a final object;
>
> (c) A has a zero-object.
>
> In that case, the zero-morphisms are exactly the identities for the group structure of the hom-sets.

Proof. Clearly, (c) implies both (a) and (b). Since the dual of a preadditive category is also preadditive, it suffices to prove that (a) implies (c). Let I be an initial object. The group $\mathrm{Hom}_{\mathsf{A}}(I, I)$ has only one element and so id_I coincides with the group identity of $\mathrm{Hom}_{\mathsf{A}}(I, I)$. Now, if $\varphi : M \to I$ is any morphism, then

$$\varphi = \mathrm{id}_I \circ \varphi = (\mathrm{id}_I + \mathrm{id}_I) \circ \varphi = \mathrm{id}_I \circ \varphi + \mathrm{id}_I \circ \varphi = \varphi + \varphi$$

and so $\mathrm{Hom}_{\mathsf{A}}(M, I)$ is the trivial group. This proves that I is also a final object. Finally, if A has a zero-object 0, then the groups $\mathrm{Hom}_{\mathsf{A}}(M, 0)$ and $\mathrm{Hom}_{\mathsf{A}}(0, N)$ are reduced to their identities and so, by the fact that composition is bilinear, the zero-morphism $M \to 0 \to N$ is the identity of $\mathrm{Hom}_{\mathsf{A}}(M, N)$. □

Observe that, in a preadditive category, two morphisms are equal if and only if their difference in the corresponding hom-set is 0. This implies that a morphism $\varphi : M \to N$ in a preadditive category is a monomorphism if and only if for all $\alpha : Z \to M$,

$$\varphi \circ \alpha = 0 \implies \alpha = 0.$$

Similarly, it is an epimorphism if and only if for all $\beta : N \to Z$,

$$\beta \circ \varphi = 0 \implies \beta = 0.$$

We are now in a position to prove a converse to the propositions 1.1.2 and 1.1.5.

Proposition 1.1.8 Let $\varphi : M \to N$ be a morphism in a preadditive category A. Then φ is a monomorphism if and only if $\ker \varphi$ is the zero-morphism $0 \to M$. Dually, φ is an epimorphism if and only if $\operatorname{coker} \varphi$ is the zero morphism $N \to 0$.

Proof. The fact that a monomorphism has the zero-morphism as its kernel was proved in proposition 1.1.2. Conversely, suppose that $0 \to M$ is a kernel for $\varphi : M \to N$, and let $\zeta : Z \to M$ be a morphism such that $\varphi \circ \zeta = 0$. The universal property implies that ζ factors through $0 \to M$ and so $\zeta = 0$, proving that φ is a monomorphism. The statement about epimorphisms follows by duality. ☐

In some sense, life is simpler in the world of modules, since finite products and coproducts coincide. Fortunately, this is already the case in preadditive categories.

Theorem 1.1.9 Let M and N be two objects in a preadditive category. Given a third object P, the following are equivalent:

(a) there exist natural projections $\pi_M : P \to M$ and $\pi_N : P \to N$ such that P satisfies the universal property of $M \times N$;

(b) there exist natural injections $\iota_M : M \to P$ and $\iota_N : N \to P$ such that P satisfies the universal property of $M \coprod N$;

(c) there exist morphisms $\pi_M : P \to M$, $\pi_N : P \to N$, $\iota_M : M \to P$ and $\iota_N : N \to P$ such that

$$\pi_M \circ \iota_M = \mathrm{id}_M, \quad \pi_N \circ \iota_N = \mathrm{id}_N, \quad \pi_M \circ \iota_N = 0, \quad \pi_N \circ \iota_M = 0,$$
$$\iota_M \circ \pi_M + \iota_N \circ \pi_N = \mathrm{id}_P .$$

Moreover, under these conditions we have that

$$\iota_M = \ker \pi_N, \quad \iota_N = \ker \pi_M, \quad \pi_M = \operatorname{coker} \iota_N, \quad \pi_N = \operatorname{coker} \iota_M.$$

If P satisfies any of the conditions above, we say that P is the direct sum $M \oplus N$.

Proof. By duality, it suffices to prove the equivalence of (a) and (c). Given (a), we use the universal property of products to obtain our desired morphisms ι_M and ι_N as the unique morphisms that satisfy $\pi_M \circ \iota_M = \mathrm{id}_M$, $\pi_N \circ \iota_N = \mathrm{id}_N$, $\pi_M \circ \iota_N = 0$ and

$\pi_N \circ \iota_M = 0$:

$$\begin{tikzcd}
& & M \\
M \arrow[urr, bend left, "\mathrm{id}_M"] \arrow[drr, bend right, "0"'] \arrow[r, dashed, "\iota_M"] & P \arrow[ur, "\pi_M"'] \arrow[dr, "\pi_N"] & \\
& & N
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & M \\
N \arrow[urr, bend left, "0"] \arrow[drr, bend right, "\mathrm{id}_N"'] \arrow[r, dashed, "\iota_N"] & P \arrow[ur, "\pi_M"'] \arrow[dr, "\pi_N"] & \\
& & N.
\end{tikzcd}$$

We then affirm that $\iota_M \circ \pi_M + \iota_N \circ \pi_N = \mathrm{id}_P$. Indeed, observe that the left-hand side satisfies

$$\pi_M \circ (\iota_M \circ \pi_M + \iota_N \circ \pi_N) = \pi_M \circ \iota_M \circ \pi_M + \pi_M \circ \iota_N \circ \pi_N = \pi_M + 0 = \pi_M$$
$$\pi_N \circ (\iota_M \circ \pi_M + \iota_N \circ \pi_N) = \pi_N \circ \iota_M \circ \pi_M + \pi_N \circ \iota_N \circ \pi_N = 0 + \pi_N = \pi_N.$$

But then both $\iota_M \circ \pi_M + \iota_N \circ \pi_N$ and id_P fit in the place of the dotted morphism which makes the diagram

$$\begin{tikzcd}
& & M \\
P \arrow[urr, bend left, "\pi_M"] \arrow[drr, bend right, "\pi_N"'] \arrow[r, dotted] & P \arrow[ur, "\pi_M"'] \arrow[dr, "\pi_N"] & \\
& & N
\end{tikzcd}$$

commute. The uniqueness part of the universal property of products then implies that they are equal, proving (c).

Now, given (c) and an object Q with morphisms $\gamma_M : Q \to M$ and $\gamma_N : Q \to N$, we need to show that there is a unique morphism $\gamma : Q \to P$ making the diagram

$$\begin{tikzcd}
& & M \\
Q \arrow[urr, bend left, "\gamma_M"] \arrow[drr, bend right, "\gamma_N"'] \arrow[r, dashed, "\gamma"] & P \arrow[ur, "\pi_M"'] \arrow[dr, "\pi_N"] & \\
& & N
\end{tikzcd}$$

commute. For the existence, we define $\gamma := \iota_M \circ \gamma_M + \iota_N \circ \gamma_N$. The diagram above then commutes since

$$\pi_M \circ \gamma = \pi_M \circ \iota_M \circ \gamma_M + \pi_M \circ \iota_N \circ \gamma_N = \gamma_M + 0 = \gamma_M,$$
$$\pi_N \circ \gamma = \pi_N \circ \iota_M \circ \gamma_M + \pi_N \circ \iota_N \circ \gamma_N = 0 + \gamma_N = \gamma_N.$$

Moreover, if $\gamma' : Q \to P$ is another morphism making the diagram commute, then,

$$\begin{aligned}\gamma' &= \mathrm{id}_P \circ \gamma' = (\iota_M \circ \pi_M + \iota_N \circ \pi_N) \circ \gamma' \\ &= \iota_M \circ \pi_M \circ \gamma' + \iota_N \circ \pi_N \circ \gamma' \\ &= \iota_M \circ \gamma_M + \iota_N \circ \gamma_N = \gamma.\end{aligned}$$

This proves (a). Assuming all the equivalent conditions for P to be the direct sum $M \oplus N$, we now show that $\iota_M = \ker \pi_N$. Since $\pi_N \circ \iota_M = 0$, it suffices to prove that if $\zeta : Z \to P$ satisfies $\pi_N \circ \zeta = 0$, then there exists a unique morphism $Z \to M$ making the diagram

$$\begin{tikzcd}
M \arrow[r, "\iota_M"] & P \arrow[r, shift left, "\pi_N"] \arrow[r, shift right, "0"'] & N \\
Z \arrow[u, dashed] \arrow[ru, "\zeta"'] & &
\end{tikzcd}$$

commute. We affirm that $\pi_M \circ \zeta$ is the desired morphism $Z \to M$. Indeed, we observe that

$$\begin{aligned}\pi_M \circ (\iota_M \circ \pi_M \circ \zeta) &= \pi_M \circ \zeta \\ \pi_N \circ (\iota_M \circ \pi_M \circ \zeta) &= 0 = \pi_N \circ \zeta\end{aligned}$$

since $\pi_M \circ \iota_M = \mathrm{id}_M$ and $\pi_N \circ \iota_M = 0$. As before, using the uniqueness part of the universal property of products, we have that $\iota_M \circ \pi_M \circ \zeta = \zeta$, proving that the diagram above commutes. This is the unique morphism making it commute because, as $\pi_M \circ \iota_M = \mathrm{id}_M$, ι_M is a monomorphism.

We can prove that $\iota_N = \ker \pi_M$ in the same way and then $\pi_M = \operatorname{coker} \iota_N$ and $\pi_N = \operatorname{coker} \iota_M$ follow by duality. □

A perk from the fact that direct sums in preadditive categories have both canonical projections and canonical injections is that it allows us to write morphisms using a matrix notation. If M_1, M_2, N_1, N_2 are four objects in a preadditive category, a morphism

$$\varphi : M_1 \oplus M_2 \to N_1 \oplus N_2$$

is completely determined by the four morphisms

$$\begin{aligned}\varphi_{11} &= \pi_1 \circ \varphi \circ \iota_1 : M_1 \to N_1 \\ \varphi_{12} &= \pi_1 \circ \varphi \circ \iota_2 : M_2 \to N_1 \\ \varphi_{21} &= \pi_2 \circ \varphi \circ \iota_1 : M_1 \to N_2 \\ \varphi_{22} &= \pi_2 \circ \varphi \circ \iota_2 : M_2 \to N_2.\end{aligned}$$

Henceforth we will represent such a morphism φ by the matrix

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}.$$

Using the correspondence between A-module morphisms $A \to A$ and elements of A, this is nothing but the matrix notation used in linear algebra to describe A-module morphisms $A^{\oplus n} \to A^{\oplus m}$. Given another morphism

$$\psi : N_1 \oplus N_2 \to P_1 \oplus P_2,$$

the matrix representation of the composition $\psi \circ \varphi$ is simply the matrix product of the individual matrices. Similarly, the sum of two morphisms $M_1 \oplus M_2 \to N_1 \oplus N_2$ is represented by the sum of the individual matrices. It is clear that this notation allows us to describe morphisms of the form

$$\bigoplus_{i=1}^{n} M_i \to \bigoplus_{j=1}^{m} N_j$$

for any positive integers n, m.

Finally, we impose the existence of zero-objects and binary products. This suffices to guarantee the existence of finite products and coproducts, which coincide by the theorem 1.1.9.

Definition 1.1.8 — Additive category. A preadditive category A is *additive* if it has a zero-object and binary products.

The prototypical example of an additive category surely is A-Mod but Ab and the category of Banach spaces with continuous linear maps are also examples of additive categories. Nevertheless, Grp is not additive since finite products and coproducts do not coincide, and neither is the category of Banach spaces with linear contractions as finite products and coproducts are not isometric.

Even though additive categories do not suffer from some of the problems we met before, they may still fail to have kernels or cokernels. For example, the category of finitely generated A-modules, when A is not noetherian, is additive but has morphisms without kernels. Furthermore, even when the additive category in consideration has kernels and cokernels, the usual first isomorphism theorem may not hold. We discuss those questions in the next section.

We finish this section with another interesting consequence of the theorem 1.1.9: the preadditive structure in an additive category is unique.

Proposition 1.1.10 Let A be a category with a zero-object and binary products. Then A has at most one abelian group structure on its hom-sets.

Proof. We endow A with any preadditive structure, and then we'll show that the addition of morphisms is actually determined by the limit-colimit structure of A.

Let $\varphi_1, \varphi_2 : M \to N$ be two morphisms in A. We define a map $\alpha : M \to M \oplus M$ by the universal property of products and a map $\beta : N \oplus N \to N$ by the universal property of coproducts:

$$\begin{tikzcd}
 & & M & & N \arrow[dr, "\iota_N"'] \arrow[drr, bend left, "\mathrm{id}_N"] & & \\
M \arrow[urr, bend left, "\mathrm{id}_M"] \arrow[r, dashed, "\alpha"] \arrow[drr, bend right, "\mathrm{id}_M"'] & M \oplus M \arrow[ur, "\pi_M"'] \arrow[dr, "\pi'_M"] & & \text{and} & & N \oplus N \arrow[r, dashed, "\beta"] & N. \\
 & & M & & N \arrow[ur, "\iota'_N"] \arrow[urr, bend right, "\mathrm{id}_N"'] & &
\end{tikzcd}$$

We observe that, by the theorem 1.1.9 and the uniqueness of the universal property of products, the equations

$$\begin{aligned}
\pi_M \circ (\iota_M + \iota'_M) &= \pi_M \circ \iota_M + \pi_M \circ \iota'_M = \mathrm{id}_M + 0 = \mathrm{id}_M = \pi_M \circ \alpha, \\
\pi'_M \circ (\iota_M + \iota'_M) &= \pi'_M \circ \iota_M + \pi'_M \circ \iota'_M = 0 + \mathrm{id}_M = \mathrm{id}_M = \pi'_M \circ \alpha
\end{aligned}$$

imply that $\alpha = \iota_M + \iota'_M$. The same exact reasoning shows that $\beta = \pi_N + \pi'_N$.

Now, we affirm that the composition $M \to M \oplus M \to N \oplus N \to N$, where the map $\psi : M \oplus M \to N \oplus N$ in the middle is given by

$$\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix},$$

is the sum $\varphi_1 + \varphi_2$. Indeed, the composition is

$$\begin{aligned}
\beta \circ \psi \circ \alpha &= (\pi_N + \pi'_N) \circ \psi \circ (\iota_M + \iota'_M) \\
&= \pi_N \circ \psi \circ \iota_M + \pi'_N \circ \psi \circ \iota_M + \pi_N \circ \psi \circ \iota'_M + \pi'_N \circ \psi \circ \iota'_M \\
&= \varphi_1 + 0 + 0 + \varphi_2 = \varphi_1 + \varphi_2
\end{aligned}$$

by the very definition of ψ. □

1.2 Abelian categories

As we saw, whenever kernels and cokernels exist, they behave reasonably well. However, their possible lack of existence prevents us from going further. Moreover, despite the fact that kernels are always monomorphisms and cokernels are always epimorphisms, there's no guarantee that every monomorphism is a kernel and that every epimorphism is a cokernel. It just so happens that demanding these properties is enough for us to have the first isomorphism theorem.

Definition 1.2.1 — Abelian category. An additive category A is *abelian* if it possesses kernels and cokernels, if every monomorphism is the kernel of some morphism and if every epimorphism is the cokernel of some morphism.

For now, our only real example of an abelian category is A-Mod and its variants, such as Ab, the category of finitely generated modules over a noetherian ring, the category of finite abelian groups, their opposites, and so forth. But the reader shouldn't worry about having few examples; a plethora of abelian categories lie ahead.

In an abelian category, every monomorphism is the kernel of some morphism. We can actually be more precise.

Proposition 1.2.1 In an abelian category A, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

Proof. Let $\varphi : M \to N$ be a monomorphism which is the kernel of some morphism $\beta : N \to Z$. Since A is abelian, φ has a cokernel $\pi : N \to C$. The universal property of the cokernel shows that β factors through π.

$$\begin{tikzcd}
 & & Z \\
M \arrow[r, "\varphi"] & N \arrow[r, "\pi"] \arrow[ru, "\beta"] & C \arrow[u, dashed]
\end{tikzcd}$$

We show that φ satisfies the universal property defining the kernel of π. Let $K \to N$ be a morphism whose composition with π is the zero-morphism.

$$\begin{tikzcd}
 & & Z \\
M \arrow[r, "\varphi"] & N \arrow[r, "\pi"] \arrow[ru, "\beta"] & C \arrow[u] \\
K \arrow[ru] \arrow[rru, bend right, "0"] & &
\end{tikzcd}$$

By the commutativity of the diagram, $K \to N \to Z$ is also the zero-morphism. But φ is the kernel of β and so there exists a unique induced morphism $K \to M$, proving our claim. The statement about epimorphisms follows by duality. ☐

This proposition implies a quick criterion for deciding when a full subcategory of an abelian category is abelian.

Corollary 1.2.2 Let A be an abelian category and let C be a full subcategory. Suppose that the zero-object of A is in C and that C is closed under binary sums, kernels, and cokernels. Then C is also abelian.

Proof. The only thing we have to verify is that every monomorphism is the kernel of some morphism and that every epimorphism is the kernel of some morphism. Now, let φ be a monomorphism in C. This implies that its kernel in C is the zero-morphism but, since kernels in C and A coincide, φ is also a monomorphism in A. We observe that, as C is closed under cokernels, $\psi := \operatorname{coker} \varphi$ is a morphism in C. Since A is abelian, the preceding proposition implies that φ satisfies the universal property of $\ker \psi$ in A and, a fortiori, in C. This proves that every monomorphism in C is the kernel of some morphism in C. The result about epimorphisms follows by duality. □

Recall that in any category, isomorphisms are both monic and epic. The converse may fail to hold even in usual categories, such as Ring, where the inclusion $\mathbb{Z} \to \mathbb{Q}$ is a monomorphism and an epimorphism but is clearly not an isomorphism. Luckily, the proposition 1.2.1 also implies that the converse holds in abelian categories.

> **Corollary 1.2.3** Let $\varphi : M \to N$ be a morphism in an abelian category A. Then φ is an isomorphism if and only if it is both a monomorphism and an epimorphism.

Proof. If φ is both monic and epic, its kernel is $0 \to M$ and its cokernel is $N \to 0$. Furthermore, by proposition 1.2.1, φ is the kernel of $N \to 0$ and the cokernel of $0 \to M$. Now consider the diagram below.

$$\begin{tikzcd}
 & & N \arrow[d, "\mathrm{id}_N"] & \\
0 \arrow[r] & M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

Since $N \to N \to 0$ is the zero morphism and φ is the kernel of $N \to 0$, we obtain a unique morphism $\psi : N \to M$ making the diagram

$$\begin{tikzcd}
 & & N \arrow[d, "\mathrm{id}_N"] \arrow[dl, dashed, "\psi"'] & \\
0 \arrow[r] & M \arrow[r, "\varphi"] & B \arrow[r] & 0
\end{tikzcd}$$

commute. As $\varphi \circ \psi = \mathrm{id}_N$, this shows that φ has a right-inverse. Similarly, the fact that φ is the cokernel of $0 \to M$ implies the existence of a unique morphism $\eta : N \to M$ such that the diagram

$$\begin{tikzcd}
 & M & & \\
0 \arrow[r] & M \arrow[r, "\varphi"] \arrow[u, "\mathrm{id}_M"] & N \arrow[r] \arrow[ul, dashed, "\eta"'] & 0
\end{tikzcd}$$

commutes. It follows that φ has both a left-inverse η and a right-inverse ψ. Thus, $\eta = \psi$ is a two-sided inverse of φ and so φ is an isomorphism. The converse holds in every category. □

We observe that this corollary implies that the category of Banach spaces with bounded linear maps is not abelian. A bounded linear map $T : X \to Y$ is a monomorphism if it's injective and an epimorphism if $\operatorname{im} T$ is dense in Y. But there exists monomorphisms with dense image which are not isomorphisms; the inclusion $\ell_1 \to \ell_2$, for example.

Earlier, we said that demanding every monomorphism to be a kernel and every epimorphism to be a cokernel is enough to guarantee the first isomorphism theorem. In order to understand how we should even enunciate such a result, we have to make sense of images in abelian categories. As with kernels and cokernels, this is best done via a suitable universal property.

Let's translate our intuitive notion of the image of a morphism $\varphi : M \to N$ in **Set** to a purely arrow-theoretic statement. The main point in **Set** is that $\operatorname{im} \varphi$ is the smallest subset of N to which we can restrict the codomain of φ to. In other words, we can factor $\varphi : M \to N$ as

$$M \longrightarrow \operatorname{im} \varphi \hookrightarrow N,$$

where $\operatorname{im} \varphi \to N$ is injective and $\operatorname{im} \varphi$ is the smallest subset of N which allows this decomposition. Switching to categorical terms, we arrive at the following universal property: the image of $\varphi : M \to N$ is a monomorphism $\iota : K \to N$ such that φ factors through ι and that is initial with these properties. That is, if $L \to N$ is another monomorphism through which φ also factors, then it exists a unique morphism $K \to L$ such that the diagram

$$\begin{tikzcd}
 & L \arrow[dr, hook] & \\
M \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[u, dashed] \arrow[r, hook, "\iota"] & N
\end{tikzcd}$$

commutes. In an arbitrary category, it could very well happen that no morphism $\iota : K \to N$ satisfies this universal property. Luckily, this is never the case in the realm of abelian categories.

Proposition 1.2.4 Let $\varphi : M \to N$ be a morphism in an abelian category, and let $\iota : K \to N$ be the kernel of $\operatorname{coker} \varphi$. Then ι is a monomorphism through which φ factors, and it is initial with these properties.

Proof. It is clear that ι is a monomorphism by the fact that it is a kernel. Since $\iota : K \to N$ is the kernel of $\operatorname{coker} \varphi : N \to C_\varphi$, the diagram

$$\begin{tikzcd}
M \arrow[r, "\varphi"] & N \arrow[r, shift left, "\operatorname{coker} \varphi"] \arrow[r, shift right, "0"'] & C_\varphi \\
K \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

commutes. The universal property of the kernel then implies the existence of a morphism $M \to K$ factoring φ through ι. We now show that ι satisfies the desired universal property. Let $\lambda : L \to N$ be another monomorphism through which φ factors, and consider its cokernel $N \to C_\lambda$.

$$\begin{tikzcd}
 & L \arrow[dr, hook, "\lambda"] & & \\
M \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[r, hook, "\iota"] & N \arrow[dr, "\operatorname{coker}\lambda"'] & \\
 & & & C_\lambda
\end{tikzcd}$$

Since φ factors through λ, the composition $M \to N \to C_\lambda$ is 0. The universal property of $\operatorname{coker}\varphi$ induces a morphism $C_\varphi \to C_\lambda$:

$$\begin{tikzcd}
 & L \arrow[dr, hook, "\lambda"] & & \\
M \arrow[ur] \arrow[r] \arrow[rr, bend right, "\varphi"'] & K \arrow[r, hook, "\iota"] & N \arrow[r, "\operatorname{coker}\varphi"] \arrow[dr, "\operatorname{coker}\lambda"'] & C_\varphi \arrow[d, dashed] \\
 & & & C_\lambda .
\end{tikzcd}$$

Observe that since $K \to N \to C_\varphi$ is the zero-morphism, so is $K \to N \to C_\lambda$. But λ is a monomorphism, which implies that it is the kernel of $\operatorname{coker}\lambda$. Its universal property then implies the existence of a unique morphism $K \to L$ making the diagram commute. □

Since all there is to know about the image of a morphism φ is encoded in the $\operatorname{im}\varphi = \ker(\operatorname{coker}\varphi)$ mantra, we use it to *define* images from now on.

Definition 1.2.2 — Image. Let $\varphi : M \to N$ be a morphism in an abelian category. Its *image*, denoted $\operatorname{im}\varphi$, is the kernel of $\operatorname{coker}\varphi$.

As it is probably clear by now, the image of a morphism $\varphi : M \to N$ of A-modules is simply the inclusion $I \to N$, where I is the set-theoretical image of φ. Indeed, $\operatorname{coker}\varphi$ is simply $N \to N/I$ and its kernel is nothing but $I \to N$.

Inverting all the arrows, we arrive at the dual notion of the image of a morphism.

Definition 1.2.3 — Coimage. Let $\varphi : M \to N$ be a morphism in an abelian category. Its *coimage*, denoted $\operatorname{coim}\varphi$, is the cokernel of $\ker\varphi$.

By duality, the proposition 1.2.4 gives a universal property for the coimage of a morphism $\varphi : M \to N$ in an abelian category: it is an epimorphism $\pi : M \to C$ such that φ factors through π and such that if $M \to D$ is another epimorphism through which φ also factors, then it exists a unique morphism $D \to C$ making the diagram

$$\begin{tikzcd}
M \arrow[r, two heads, "\pi"'] \arrow[rr, bend left, "\varphi"] \arrow[dr, two heads] & C \arrow[r] & N \\
& D \arrow[u, dashed] \arrow[ur] &
\end{tikzcd}$$

commute.

Now, our sought-for first isomorphism theorem is simply a particular relation between the image and the coimage of a given morphism. In A-Mod, the coimage of a morphism $\varphi : M \to N$ is the quotient map $M \to M/K$, where K is the set-theoretical kernel of φ. The first isomorphism theorem in this context amounts to the fact that we can factor $\varphi : M \to N$ as

$$\begin{tikzcd}
M \arrow[r, "\operatorname{coim} \varphi"'] \arrow[rrr, bend left, "\varphi"] & M/K \arrow[r] & I \arrow[r, "\operatorname{im} \varphi"'] & N,
\end{tikzcd}$$

where the morphism in the middle, induced by φ, is an isomorphism. In this form, the result holds in arbitrary abelian categories.

Theorem 1.2.5 — First isomorphism theorem. Let $\varphi : M \to N$ be a morphism in an abelian category. Then φ can be decomposed as

$$\begin{tikzcd}
M \arrow[r] \arrow[rrr, bend left, "\varphi"] & C \arrow[r, "\sim"] & K \arrow[r] & N,
\end{tikzcd}$$

where $M \to C$ is the coimage of φ, $K \to N$ is its image and $C \to K$ is an isomorphism.

Proof. The universal properties of the image and of the coimage give two decompositions of φ as follows:

$$\begin{tikzcd}
& & K \arrow[dr, hook, "\operatorname{im} \varphi"] & \\
M \arrow[urr, "\alpha"] \arrow[rrr, "\varphi"] \arrow[dr, two heads, "\operatorname{coim} \varphi"'] & & & N. \\
& C \arrow[urr, "\beta"'] & &
\end{tikzcd}$$

In order to use the universal property of $\operatorname{im} \varphi$ to obtain an induced morphism $K \to C$, we must prove that β is a monomorphism. (Similarly, we could prove that α is an epimorphism and use the universal property of $\operatorname{coim} \varphi$.) Since every monomorphism is the kernel of its cokernel, $\ker(\operatorname{coim} \beta) = \ker(\operatorname{coker}(\ker \beta)) = \ker \beta$. It suffices then to show that $\ker(\operatorname{coim} \beta) = 0$. We observe that the composition of $\operatorname{coim} \beta$ and $\operatorname{coim} \varphi$

$$\begin{tikzcd}
M \arrow[r, two heads, "\operatorname{coim} \varphi"] \arrow[rrr, bend right, "\varphi"'] & C \arrow[r, two heads, "\operatorname{coim} \beta"'] \arrow[rr, bend left, "\beta"] & N' \arrow[r] & N
\end{tikzcd}$$

is an epimorphism through which φ factors. The universal property of $\operatorname{coim} \varphi$ then implies that $\operatorname{coim} \beta$ is an isomorphism, concluding that $\ker \beta = 0$ and so β is a monomorphism.

As we said above, the universal property of $\operatorname{im} \varphi$ induces a morphism $\psi : K \to C$ making the diagram

$$\begin{tikzcd}
 & & K \arrow[dr, hook, "\operatorname{im} \varphi"] \arrow[ddl, dashed, "\psi"'] & \\
M \arrow[urr, two heads, "\alpha"] \arrow[dr, two heads, "\operatorname{coim} \varphi"'] & & & N \\
 & C \arrow[urr, hook, "\beta"] & &
\end{tikzcd}$$

commute. Since $\beta \circ \psi = \operatorname{im} \varphi$ is a monomorphism, so is ψ. Similarly, the fact that $\psi \circ \alpha = \operatorname{coim} \varphi$ is an epimorphism implies that ψ has the same property. It follows that ψ is an isomorphism, and so it suffices to consider its inverse to be our desired morphism $C \to K$. ☐

As we'll see, this theorem even gives an alternative definition of abelian category. For now, suppose that $\varphi : M \to N$ is a morphism in an additive category that possesses kernels and cokernels. In this context, it is *not* true that $\ker(\operatorname{coker} \varphi) : K \to N$ satisfies the universal property of the image of φ[1] but, since $(\operatorname{coker} \varphi) \circ \varphi = 0$, the universal property of kernels implies that φ factors through $\ker(\operatorname{coker} \varphi)$.

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[dr, dashed] & N \\
 & K \arrow[u, "\ker(\operatorname{coker} \varphi)"']
\end{tikzcd}$$

Similarly, the universal property of cokernels implies that $M \to K$ factors through

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[1] For a counterexample, consider the morphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 in the category of torsion-free abelian groups. The reader may verify that this is an additive category, with kernels and cokernels, and that $\ker(\operatorname{coker} \varphi) = \operatorname{id} : \mathbb{Z} \to \mathbb{Z}$. Then φ is another monomorphism through which φ factors, but there's no morphism induced by the universal property of images.

$\operatorname{coker}(\ker \varphi) : M \to C$ via a morphism $\overline{\varphi} : C \to K$.

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\operatorname{coker}(\ker \varphi)"'] & N \\
C \arrow[r, dashed, "\overline{\varphi}"] & K \arrow[u, "\ker(\operatorname{coker} \varphi)"']
\end{tikzcd}$$

Our previous theorem shows that $\overline{\varphi}$ is an isomorphism whenever we're dealing with an abelian category. We affirm that this property also suffices to define an abelian category.

> **Proposition 1.2.6** Let A be an additive category that possesses kernels and cokernels. Then A is abelian if and only if for every morphism $\varphi : M \to N$, the induced morphism $\overline{\varphi} : C \to K$ is an isomorphism.

Proof. One direction was shown in the previous theorem. Conversely, suppose that $\varphi : M \to N$ is a monomorphism. Then $\ker \varphi = 0$ and so φ factors as

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_M"'] & N \\
M \arrow[r, "\overline{\varphi}"] & K. \arrow[u, "\ker(\operatorname{coker} \varphi)"']
\end{tikzcd}$$

This implies that φ satisfies the universal property of $\ker(\operatorname{coker} \varphi)$. (Since φ and $\ker(\operatorname{coker} \varphi)$ define the same subobjects of N.) By duality, it follows that every epimorphism is a cokernel. ☐

1.3 Unions and intersections

Let M be an object in a (not necessarily abelian) category A. As we saw in the beginning of this chapter, a subobject of M is an equivalence class of monomorphisms $s : S \to M$. Given another subobject defined by $t : T \to M$, we say that s is *smaller than* t if there exists a morphism $S \to T$, automatically monic, making the diagram

$$\begin{tikzcd}
S \arrow[rr, hook] \arrow[dr, hook, "s"'] & & T \arrow[dl, hook', "t"] \\
& M &
\end{tikzcd}$$

commute. This is independent of the representatives chosen for each equivalence class. Also, the morphism $S \to T$ is unique whenever it exists. This endows the collection of all subobjects of M with the structure of a partially ordered class.[2] In particular, we are able to define the union and the intersection of a family of subobjects.

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[2] It need not be a set, even when the category in question is abelian. We say that a category is *well-powered* if the subobjects of every object constitute a set.

Definition 1.3.1 Let M be an object of a category A. The *union*, if it exists, of a family of subobjects of M is their supremum in the partially ordered class of subobjects. Similarly, the *intersection* of a family of subobjects is their infimum.

We'll often use the customary symbols $\cup$ and $\cap$ to denote the union and the intersection of subobjects, leaving their target implicit.

In A-Mod, the union of two submodules S and T of a given module M is simply their sum $S + T$. In other words, it's the image of the canonical morphism $S \oplus T \to M$, which sends (s, t) to $s + t$. This description generalizes to arbitrary abelian categories.

Proposition 1.3.1 Let A be an abelian category and $S_i \to M$ be a finite collection of subobjects. The union of those subobjects exists and is given by the image of the natural map $\bigoplus_i S_i \to M$.

Proof. Factoring each $S_i \to M$ through the coproduct and then factoring the resulting morphism through its image we obtain the diagram below.

$$\begin{tikzcd}
& S_i \arrow[dl] \arrow[dr, hook] & \\
\bigoplus_i S_i \arrow[r] & K \arrow[r, hook] & M
\end{tikzcd}$$

In particular, $K \to M$ is a subobject which is greater than all of the $S_i \to M$. Now, suppose that $T \to M$ is another subobject through which all the $S_i \to M$ factor. The universal property of coproducts induces a dashed morphism making the diagram

$$\begin{tikzcd}
& S_i \arrow[d] \arrow[ddl] \arrow[ddr, hook] & \\
& T \arrow[dr, hook] & \\
\bigoplus_i S_i \arrow[ur, dashed] \arrow[r] & K \arrow[r, hook] & M
\end{tikzcd}$$

commute. (The lower triangle commutes by the unicity of the induced morphism $\bigoplus_i S_i \to M$.) Finally, the universal property of images induces a morphism $K \to T$, proving that $K \to M$ is indeed the supremum of the $S_i \to M$. $\square$

The same proof shows that the preceding description also works for infinite unions, replacing the direct sums by coproducts, whenever those coproducts exist in our category.

In a wide range of cases, the proposition below describes binary intersections.

Proposition 1.3.2 Let A be a category with pullbacks. The intersection of two subobjects $S \to M$ and $T \to M$ exists and is given by their pullback.

Proof. We recall that, in absolute generality, pullbacks preserve monomorphisms. [3, Proposition 2.5.3] That is, if

$$\begin{tikzcd} P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\ S \arrow[r, "s"] & M \end{tikzcd}$$

is a cartesian diagram and s is a monomorphism, then so is s'. Similarly for t and t', of course. In particular, $P \to M$ is a subobject which is less than $S \to M$ and $T \to M$. Moreover, $P \to M$ is their infimum, due to the universal property of pullbacks. □

Once again, the same proof shows that the intersection of a family of subobjects $S_i \to M$ exists and is given by the limit of the diagram constituted of those morphisms, as long as such limit exists.

Fortunately, abelian categories possess pullbacks[3] and they have simple descriptions. In A-Mod, the pullback of two morphisms $\varphi : M \to P$ and $\psi : N \to P$ is given by submodule of $M \oplus N$ determined by the elements (m, n) satisfying $\varphi(m) = \psi(n)$. Basically the same description works more generally. In particular, the collection of subobjects of every object in an abelian category form a lattice.

Proposition 1.3.3 Let $s : S \to M$ and $t : T \to M$ be two morphisms in an abelian category A. The kernel of the morphism

$$(s, -t) : S \oplus T \to M$$

satisfies the universal property of the pullback $S \times_M T$. Dually, if $s' : N \to S$ and $t' : N \to T$ are two morphisms in A, the cokernel of

$$\begin{pmatrix} s' \\ -t' \end{pmatrix} : N \to S \oplus T$$

satisfies the universal property of the pushout $S \coprod_N T$.

Proof. Let $\pi_S : S \oplus T \to S$ and $\pi_T : S \oplus T \to T$ be the canonical projections. Moreover, denote the kernel of $(s, -t)$ by $\kappa : P \to S \oplus T$, and pose $s' := \pi_T \circ \kappa$, $t' := \pi_S \circ \kappa$. Being more precise, the first statement is that the square

$$\begin{tikzcd} P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\ S \arrow[r, "s"'] & M \end{tikzcd}$$

---

[3] Abelian categories are even finitely complete and finitely cocomplete, since all finite limits can be constructed from terminal objects, pullbacks and equalizers. [3, Proposition 2.8.2]

is cartesian. We observe that

$$(s, -t) = \mathrm{id}_{S \oplus T} \circ (s, -t) = \begin{pmatrix} \pi_S \\ \pi_T \end{pmatrix} \circ (s, -t) = s \circ \pi_S - t \circ \pi_T.$$

This implies the commutativity of the square above, given that

$$s \circ t' - t \circ s' = s \circ \pi_S \circ \kappa - t \circ \pi_T \circ \kappa = (s, -t) \circ \kappa = 0.$$

We now prove that the square satisfies the universal property of pullbacks. Let $\varphi : Q \to S$ and $\psi : Q \to T$ be such that $s \circ \varphi = t \circ \psi$. Since

$$(s, -t) \circ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = s \circ \varphi - t \circ \psi = 0,$$

the universal property of kernels gives a unique morphism $\mu : Q \to P$ making the diagram

$$\begin{tikzcd} Q \arrow[r, "\mu"] \arrow[dr, "\begin{pmatrix} \varphi \\ \psi \end{pmatrix}"'] & P \arrow[d, "\kappa"] \\ & S \oplus T \end{tikzcd}$$

commute. Moreover, we have that

$$s' \circ \mu = \pi_T \circ \kappa \circ \mu = \pi_T \circ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \psi$$

and, similarly, that $t' \circ \mu = \varphi$. The unicity of these factorizations follows from the unicity in the universal properties of kernels and of products. As usual, the other statement follows by duality. □

As we saw in the proof of proposition 1.3.2, pullbacks preserve monomorphisms. Dually, pushouts preserve epimorphisms. In abelian categories we have even more.

Corollary 1.3.4 Let A be an abelian category. Suppose that

$$\begin{tikzcd} P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\ S \arrow[r, "s"] & M \end{tikzcd}$$

is a cartesian diagram in A, and that s is an epimorphism. Then s' is also an epimorphism and the square is also a pushout. Dually, the pushout of a monomorphism is a monomorphism, and the corresponding square is also a pullback.

Proof. We keep the same notations as in the proof of the previous proposition. We begin by proving that $(s, -t)$ is an epimorphism. Let $\rho : M \to N$ be a morphism such that $\rho \circ (s, -t) = 0$. Then, denoting by $\iota_S : S \to S \oplus T$ the natural injection, we have that

$$0 = \rho \circ (s, -t) \circ \iota_S = \rho \circ (s \circ \pi_S - t \circ \pi_T) \circ \iota_S = \rho \circ s.$$

This implies that $\rho = 0$, for s is an epimorphism. In particular, $(s, -t) = \operatorname{coker} \kappa$ due to proposition 1.2.1.

Now, let $\sigma : T \to Z$ be a morphism such that $\sigma \circ s' = 0$. Since $s' = \pi_T \circ \kappa$, the universal property of cokernels gives a morphism $\zeta : M \to Z$ making the diagram

$$\begin{tikzcd}
P \arrow[r, "\kappa"] \arrow[dr, "s'"'] & S \oplus T \arrow[r, "{(s,-t)}"] \arrow[d, "\pi_T"] & M \arrow[d, dashed, "\zeta"] \\
& T \arrow[r, "\sigma"] & Z
\end{tikzcd}$$

commute. But the equation

$$\zeta \circ s = \zeta \circ (s, -t) \circ \iota_S = \sigma \circ \pi_T \circ \iota_S = 0$$

implies that $\zeta = 0$, since s is an epimorphism. Finally, the fact that π_T is an epimorphism and satisfies $\sigma \circ \pi_T = 0$ implies that $\sigma = 0$, proving that s' is an epimorphism as well.

We now show that our cartesian square is also cocartesian. Let $\eta : S \to Q$ and $\lambda : T \to Q$ be two morphisms making the diagram

$$\begin{tikzcd}
P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \arrow[ddr, bend left, "\lambda"] & \\
S \arrow[r, "s"'] \arrow[drr, bend right, "\eta"'] & M & \\
& & Q
\end{tikzcd}$$

commute. Observe that, by the universal property of pullbacks, there exists a dashed morphism making the diagram

$$\begin{tikzcd}
K \arrow[drr, bend left, "0"] \arrow[dr, dashed] \arrow[ddr, bend right, "\ker s"'] & & & \\
& P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \arrow[ddr, bend left, "\lambda"] & \\
& S \arrow[r, "s"'] \arrow[drr, bend right, "\eta"'] & M & \\
& & & Q
\end{tikzcd}$$

commute. This implies that $\eta \circ \ker s = 0$, and so the universal properties of cokernels (since s is the cokernel of $\ker s$) gives a morphism $\xi : M \to Q$ satisfying $\eta = \xi \circ s$. Moreover, we have that

$$\lambda \circ s' = \eta \circ t' = \xi \circ s \circ t' = \xi \circ t \circ s'.$$

It follows that $\lambda = \xi \circ t$, since s' is an epimorphism. In other words, ξ makes the diagram

$$\begin{tikzcd}
P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \arrow[ddr, bend left, "\lambda"] & \\
S \arrow[r, "s"'] \arrow[drr, bend right, "\eta"'] & M \arrow[dr, "\xi"] & \\
& & Q
\end{tikzcd}$$

commute. Such a morphism is unique, due to s being an epimorphism. We conclude the result. The other statement follows by duality. ☐

Given two subobjects $S \to M$ and $T \to M$, we can naturally form the commutative diagram below.

$$\begin{tikzcd}
S \cap T \arrow[r, hook] \arrow[d, hook] & T \arrow[d, hook] \\
S \arrow[r, hook] & S \cup T
\end{tikzcd}$$

Since we can always complete this square into a diagram of the form

$$\begin{tikzcd}
S \cap T \arrow[r, hook] \arrow[d, hook] & T \arrow[d, hook] \arrow[ddr, hook, bend left] & \\
S \arrow[r, hook] \arrow[drr, hook, bend right] & S \cup T \arrow[dr, hook] & \\
& & M,
\end{tikzcd}$$

the proposition 1.3.2 gives that our original square is always cartesian. The same reasoning, along with the preceding corollary, implies that it's also cocartesian. The fact that this square is, at the same time, a pullback and a pushout is usually phrased as the motto *binary unions in abelian categories are effective*. In other words, to define a morphism $S \cup T \to P$, it suffices to find morphisms $S \to P$ and $T \to P$ which agree on the intersection $S \cap T$.

A final interesting result, which will be the soul of the next few propositions, can also be proved using the same circle of ideas.

Proposition 1.3.5 Let A be an abelian category. Given a commutative square

$$\begin{tikzcd}
P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\
S \arrow[r, "s"] & M
\end{tikzcd}$$

in A, consider the induced morphisms $k : K' \to K$ and $c : C' \to C$ between the kernels and cokernels of s' and s:

$$\begin{tikzcd}
K' \arrow[r, "\ker s'"] \arrow[d, "k"'] & P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[r, "\operatorname{coker} s'"] \arrow[d, "t"] & C' \arrow[d, "c"] \\
K \arrow[r, "\ker s"'] & S \arrow[r, "s"'] & M \arrow[r, "\operatorname{coker} s"'] & C,
\end{tikzcd}$$

If the original square is cartesian, then k is an isomorphism. Dually, if the original square is cocartesian, then c is an isomorphism.

Proof. Suppose that our square is cartesian, for the other statement follows by duality. The universal property of pullbacks gives a dashed morphism, making the diagram

$$\begin{tikzcd}
K \arrow[rrd, bend left, "0"] \arrow[rd, dashed, "p"] \arrow[rdd, bend right, "\ker s"'] & & \\
& P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\
& S \arrow[r, "s"'] & M
\end{tikzcd}$$

commute. Then, since $K \to P \to T$ is zero, the universal property of kernels gives a dashed morphism $k' : K \to K'$ making the diagram

$$\begin{tikzcd}
K' \arrow[r, "\ker s'"] & P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\
K \arrow[u, dashed, "k'"] \arrow[ru, "p"] \arrow[r, "\ker s"'] & S \arrow[r, "s"'] & M
\end{tikzcd}$$

commute. Checking the commutativity of the previous diagrams, we remark that

$$\begin{aligned}
t' \circ (\ker s') \circ k' \circ k &= t' \circ p \circ k = (\ker s) \circ k = t' \circ (\ker s') \\
s' \circ (\ker s') \circ k' \circ k &= s' \circ p \circ k = 0 \circ k = s' \circ (\ker s').
\end{aligned}$$

In other words, $\ker s'$ and $(\ker s') \circ k' \circ k$ are two morphisms making the diagram

$$\begin{tikzcd}
K' \arrow[dr, shift left, "\ker s'"] \arrow[dr, shift right, "(\ker s') \circ k' \circ k"'] & & \\
& P \arrow[r, "s'"] \arrow[d, "t'"'] & T \arrow[d, "t"] \\
& S \arrow[r, "s"'] & M
\end{tikzcd}$$

commute. The uniqueness in the universal property of pullbacks implies that they're equal. Since $\ker s'$ is a monomorphism, $k' \circ k = \mathrm{id}_{K'}$. Furthermore,

$$(\ker s) \circ k \circ k' = t' \circ (\ker s') \circ k' = \ker s.$$

As $\ker s$ is monic, we have $k \circ k' = \mathrm{id}_K$; proving that k is an isomorphism. $\square$

We basically defined an abelian category in order to have the first isomorphism theorem. Somewhat surprising, all the other isomorphism theorems are also true in this generality.

If $t : T \to M$ is a subobject, we'll denote the target of $\operatorname{coker} t$ by M/T, as it would be in A-**Mod**. We remark that if $S \to M$ is a subobject containing t, then S/T is naturally a subobject of M/T. That is, there exists a dashed monomorphism making the diagram

$$\begin{tikzcd}
& & T \arrow[dl, hook'] \arrow[dr] & & \\
& S \arrow[rr, hook] \arrow[dl] & & M \arrow[dr] & \\
S/T \arrow[rrrr, hook, dashed] & & & & M/T
\end{tikzcd}$$

commute. Indeed, the universal property of the cokernel on the left gives the existence, and the universal property of the cokernel on the right implies that the trapezoid above is a pushout; proving that the dashed morphism is monic.

Proposition 1.3.6 Let $t : T \to M$ be a subobject in an abelian category. Then,

$$\begin{aligned} u : \{\text{subobjects of } M \text{ containing } t\} &\to \{\text{subobjects of } M/T\} \\ (S \to M) &\mapsto (S/T \to M/T) \end{aligned}$$

is a lattice isomorphism. Moreover, if $S \to M$ is a subobject containing T, the objects

$$(M/T)/(S/T) \qquad \text{and} \qquad M/S$$

are isomorphic.

Before we begin the proof, recall that a partially ordered class may be seen naturally as a category. In this context, a lattice is a partially ordered class with binary products and coproducts. Similarly, a morphism of lattices can be seen as a functor preserving such (co)products.

Proof. In order to prove that u is an isomorphism of lattices, we define an inverse. Consider the function v, which sends a subobject $Q \to M/T$ to the top arrow in the pullback

$$\begin{tikzcd}
P \arrow[r, hook] \arrow[d] & M \arrow[d] \\
Q \arrow[r, hook] & M/T.
\end{tikzcd}$$

Since $T \to M \to M/T$ is zero, the universal property of pullbacks gives a dashed morphism making the diagram

$$\begin{tikzcd}
T \arrow[rrd, hook, bend left, "t"] \arrow[rd, dashed] \arrow[rdd, bend right, "0"'] & & \\
& P \arrow[r, hook] \arrow[d] & M \arrow[d, "\text{coker } t"] \\
& Q \arrow[r, hook] & M/T
\end{tikzcd}$$

commute, proving that $P \to M$ contains $T \to M$. It's clear that both u and v are order-preserving. In other words, they are functors. Applying u to the subobject $P \to M$, we obtain the commutative diagram below.

$$\begin{tikzcd}
& T \arrow[ld, hook] \arrow[rd, hook] & \\
P \arrow[rr, hook] \arrow[rd, two heads] \arrow[dd, two heads] & & M \arrow[dd, two heads] \\
& P/T \arrow[rd, hook] & \\
Q \arrow[rr, hook] & & M/T
\end{tikzcd}$$

Observe that the morphism $T \to P \to Q \to M/T$ is zero, due to the commutativity of the diagram. Actually, $T \to P \to Q$ is already zero, as $Q \to M/T$ is monic. Then, the universal property of cokernels gives a morphism $P/T \to Q$ making the diagram above commute. This morphism is both a monomorphism and an epimorphism, by the commutativity of the triangles on its sides. In other words, $u \circ v$ is the identity functor.

Now, let $S \to M$ be a subobject containing $t : T \to M$. As we say when defining the map $S/T \to M/T$, the commutative square

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d, two heads] & M \arrow[d, two heads] \\
S/T \arrow[r, hook] & M/T
\end{tikzcd}$$

is cocartesian. The corollary 1.3.4 implies that it's also cartesian, proving that $v \circ u$ is also the identity functor. Since u is an equivalence of categories, it preserves products and coproducts. In particular, it's an isomorphism of lattices.

The isomorphism between $(M/T)/(S/T)$ and M/S follows from the proposition 1.3.5, applied to the cocartesian square above. ☐

The last isomorphism theorem also follows from the machinery developed in this section.

Proposition 1.3.7 Let $S \to M$ and $T \to M$ be subobjects in an abelian category. Then the objects

$$(S \cup T)/T \qquad \text{and} \qquad S/(S \cap T)$$

are isomorphic.

Proof. Since binary unions in abelian categories are effective, the commutative diagram

$$\begin{tikzcd}
S \cap T \arrow[r, hook] \arrow[d, hook] & T \arrow[d, hook] \\
S \arrow[r, hook] & S \cup T
\end{tikzcd}$$

is a pushout. The result then follows from the same proposition 1.3.5. ☐

1.4 Exactness in abelian categories

After all this foundational work, we can at long last understand how exact sequences work in an abelian category.

Definition 1.4.1 — Exact sequence. Consider a sequence of objects and morphisms in an abelian category:

$$\cdots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow \cdots .$$

We say that this sequence is *exact* at N if $\ker \psi$ and $\operatorname{im} \varphi$ define the same subobject of N. It is *exact* if it's exact at every object.

As it is the case in A-Mod, most properties about morphisms can be stated in terms of exact sequences. For example,

$$0 \longrightarrow M \xrightarrow{\varphi} N$$

is an exact sequence if and only if φ is a monomorphism. Likewise,

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

is an exact sequence if and only if φ is a kernel of ψ. Also,

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0.$$

is exact if and only if φ is a kernel of ψ and ψ is cokernel of φ. These last exact sequences are so important that they deserve a name.

Definition 1.4.2 — Short exact sequence. An exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0.$$

is said to be a *short exact sequence*.

Another reason for the importance of short exact sequences is that we can check the exactness of an arbitrary sequence by intertwining it with short exact sequences. Let's illustrate this procedure with a sequence of the form

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} M_4.$$

Using the theorem 1.2.5, we can enlarge our diagram to be

$$\begin{tikzcd}
 & & C_2 \arrow[dr, "\operatorname{im}\varphi_1"] & & & & C_4 \arrow[dr, "\operatorname{im}\varphi_3"] & & \\
 & M_1 \arrow[ur, "\operatorname{coim}\varphi_1"] \arrow[rr, "\varphi_1"] & & M_2 \arrow[rr, "\varphi_2"] \arrow[dr, "\operatorname{coim}\varphi_2"'] & & M_3 \arrow[ur, "\operatorname{coim}\varphi_3"] \arrow[rr, "\varphi_3"] & & M_4 \arrow[dr, "\operatorname{coker}\varphi_3"] & \\
C_1 \arrow[ur, "\ker\varphi_1"] & & & & C_3 \arrow[ur, "\operatorname{im}\varphi_2"'] & & & & C_5.
\end{tikzcd}$$

Using that kernels and images are monic and that cokernels and coimages are epic, we obtain a yet larger diagram which is exact at all the C_i, at M_1, and at M_4.

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
 & & & C_2 \arrow[ur] \arrow[dr, "\operatorname{im}\varphi_1"] & & & & C_4 \arrow[ur] \arrow[dr, "\operatorname{im}\varphi_3"] & & & \\
 & & M_1 \arrow[ur, "\operatorname{coim}\varphi_1"] \arrow[rr, "\varphi_1"] & & M_2 \arrow[rr, "\varphi_2"] \arrow[dr, "\operatorname{coim}\varphi_2"'] & & M_3 \arrow[ur, "\operatorname{coim}\varphi_3"] \arrow[rr, "\varphi_3"] & & M_4 \arrow[dr, "\operatorname{coker}\varphi_3"] & & \\
 & C_1 \arrow[ur, "\ker\varphi_1"] & & & & C_3 \arrow[ur, "\operatorname{im}\varphi_2"'] \arrow[dr] & & & & C_5 \arrow[dr] & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

Now, we affirm that our original sequence is exact if and only if those four diagonal sequences are exact. Indeed, the only place where the diagonal sequences could lack exactness is at M_2 and M_3. Being exact at M_2 means that $\ker(\operatorname{coim}\varphi_2) = \operatorname{im}(\operatorname{im}\varphi_1)$ which is equivalent to $\ker\varphi_2 = \operatorname{im}\varphi_1$. The same holds for exactness at M_3, and it's clear that this procedure generalizes to sequences of arbitrary length.

A particularly frequent kind of short exact sequence appears when we consider the direct sum of two objects M and N. Since $M \oplus N$ fulfills both the role of the product and the coproduct of M and N, we have a natural injection $\iota : M \to M \oplus N$ and a natural projection $\pi : M \oplus N \to N$. These objects fit nicely into a sequence

$$0 \longrightarrow M \xrightarrow{\iota} M \oplus N \xrightarrow{\pi} N \longrightarrow 0,$$

which is exact since ι is the kernel of π and π is the cokernel of ι. (Theorem 1.1.9.) This is the prototypical example of a split exact sequence.

Definition 1.4.3 — Split exact sequence. A short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

is said to *split* if there's a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\sim"] & N \arrow[r] \arrow[d, "\sim"] & P \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & M' \arrow[r, "\iota"] & M' \oplus P' \arrow[r, "\pi"] & P' \arrow[r] & 0
\end{tikzcd}$$

in which all the vertical maps are isomorphisms, ι is the natural injection and π is the natural projection.

Understanding which exact sequences are split will allow us to understand injective and projective objects better, to understand when a morphism has a right- or left-inverse, and to gain a refined version of the first isomorphism theorem. The following theorem takes care of these last two tasks.

Theorem 1.4.1 — Splitting lemma. A short exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P \longrightarrow 0$$

is split if and only if one of the conditions below is satisfied:

(a) there exists a morphism $\sigma : P \to N$ such that $\psi \circ \sigma = \mathrm{id}_P$;

(b) there exists a morphism $\rho : N \to M$ such that $\rho \circ \varphi = \mathrm{id}_M$.

Proof. If the sequence is split, then by composing the natural injections/projections with the vertical maps in the definition 1.4.3 we obtain the desired morphisms $\sigma : P \to N$ and $\rho : N \to M$.

Conversely, we suppose that (a) holds and prove that the sequence is split. Our approach will be based on the construction of a morphism $\rho : N \to M$ as in (b) such that

$$\rho \circ \varphi = \mathrm{id}_M, \quad \psi \circ \sigma = \mathrm{id}_P, \quad \rho \circ \sigma = 0, \quad \psi \circ \varphi = 0,$$
$$\varphi \circ \rho + \sigma \circ \psi = \mathrm{id}_N .$$

This is enough for the theorem 1.1.9 to imply that N is isomorphic to the direct sum of M and P. We already have two of the equations: $\psi \circ \sigma = \mathrm{id}_P$ and $\psi \circ \varphi = 0$.

In order to find a morphism ρ such that $\varphi \circ \rho + \sigma \circ \psi = \mathrm{id}_N$, we consider the morphism $\mathrm{id}_N - \sigma \circ \psi$. Observe that

$$\psi \circ (\mathrm{id}_N - \sigma \circ \psi) = \psi - \underbrace{\psi \circ \sigma}_{\mathrm{id}_P} \circ \psi = 0.$$

The universal property of kernels, by the fact that $\varphi = \ker \psi$, implies the existence of a unique morphism $\rho : N \to M$ such that $\varphi \circ \rho = \mathrm{id}_N - \sigma \circ \psi$, proving another equation.

Finally, we observe that, since φ is a monomorphism,

$$\varphi \circ \rho \circ \varphi = (\mathrm{id}_N - \sigma \circ \psi) \circ \varphi = \varphi - \sigma \circ \underbrace{\psi \circ \varphi}_{0} = \varphi$$

implies that $\rho \circ \varphi = \mathrm{id}_M$. Similarly,

$$\varphi \circ \rho \circ \sigma = (\mathrm{id}_N - \sigma \circ \psi) \circ \sigma = \sigma - \sigma \circ \underbrace{\psi \circ \sigma}_{\mathrm{id}_P} = 0$$

and so $\rho \circ \sigma = 0$, proving the last equation.

The proof that (b) implies that the sequence is split is basically the same. ☐

As promised, the splitting lemma gives a necessary and sufficient condition for a morphism to have a right- or left-inverse. We recall that a morphism that has a right-inverse is necessarily an epimorphism and that a morphism that has a left-inverse is necessarily a monomorphism.

Corollary 1.4.2 Let $\varphi : M \to N$ be a morphism in an abelian category. Then φ has a left-inverse if and only if the sequence

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\operatorname{coker} \varphi} C \longrightarrow 0$$

is split, and it has a right-inverse if and only if the sequence

$$0 \longrightarrow K \xrightarrow{\ker \varphi} M \xrightarrow{\varphi} N \longrightarrow 0$$

is split.

The splitting lemma also provides a refinement of the first isomorphism theorem. For that, we observe that a morphism $\varphi : M \to N$ determines a sequence

$$0 \longrightarrow K \xrightarrow{\ker \varphi} M \xrightarrow{\operatorname{coim} \varphi} I \longrightarrow 0,$$

which is exact since $\ker \varphi$ is the kernel of $\operatorname{coim} \varphi = \operatorname{coker}(\ker \varphi)$ (every monomorphism is the kernel of its cokernel) and $\operatorname{coim} \varphi$ is the cokernel of $\ker \varphi$. We also recall that, due to the first isomorphism theorem, I is isomorphic to the source of $\operatorname{im} \varphi$.

Corollary 1.4.3 Let $\varphi : M \to N$ be a morphism in an abelian category, let $\ker \varphi : K \to M$ be its kernel and $\operatorname{coim} \varphi : M \to I$ be its coimage. If there exists a morphism $\sigma : I \to M$ such that $(\operatorname{coim} \varphi) \circ \sigma = \mathrm{id}_I$ or a morphism $\rho : M \to K$ such that $\rho \circ \ker \varphi = \mathrm{id}_K$, then $M \cong K \oplus I$.

In the category of finite-dimensional vector spaces over a field, this result holds unconditionally, since two such vector spaces are isomorphic if and only if they have the same dimension. Thus, this corollary follows from the rank-nullity theorem. But, in general abelian categories, the decomposition $M \cong K \oplus I$ need not hold.[4]

1.5 Functors on abelian categories

Just as all the useful morphisms on a group must preserve its structure, so must the useful functors on a preadditive category.

---

[4] Just take the projection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, for example.

Definition 1.5.1 — Additive functor. Let A and B be two preadditive categories. A functor $F : \mathsf{A} \to \mathsf{B}$ is said to be *additive* if, for all objects M, N in A, the induced map

$$\begin{aligned} \mathrm{Hom}_{\mathsf{A}}(M, N) &\to \mathrm{Hom}_{\mathsf{B}}(F(M), F(N)) \\ \varphi &\mapsto F(\varphi) \end{aligned}$$

is a morphism of groups.

Basically all the functors defined between preadditive categories that we'll encounter are additive. Some examples are $\mathrm{Hom}_{\mathsf{A}}(M, -)$ and, in A-Mod, the tensor product functor $M \otimes_A -$.

There's an interesting criterion for a functor to be additive. For that, we observe that if $F : \mathsf{A} \to \mathsf{B}$ is a functor between additive categories and M, N are two objects of A, then the universal property of products induces a morphism $F(M \oplus N) \to F(M) \oplus F(N)$:

$$\begin{tikzcd} & & F(M) \\ F(M \oplus N) \arrow[r, dashed] \arrow[urr, bend left, "F(\pi_M)"] \arrow[drr, bend right, "F(\pi_N)"'] & F(M) \oplus F(N) \arrow[ur] \arrow[dr] & \\ & & F(N), \end{tikzcd}$$

where $F(M) \oplus F(N) \to F(M)$ and $F(M) \oplus F(N) \to F(N)$ are the natural projections. Similarly, the universal property of coproducts induces a morphism $F(M) \oplus F(N) \to F(M \oplus N)$.

Proposition 1.5.1 Let $F : \mathsf{A} \to \mathsf{B}$ be a functor between additive categories. Then the following are equivalent:

(a) F is additive;

(b) the natural map $F(M) \oplus F(N) \to F(M \oplus N)$ is an isomorphism for every M, N in A;

(c) the natural map $F(M \oplus N) \to F(M) \oplus F(N)$ is an isomorphism for every M, N in A.

Proof. Due to the fact that an additive functor preserves composition and addition of morphisms, the theorem 1.1.9 gives automatically that (a) implies (b) and (c). Also, (b) and (c) are equivalent since the uniqueness part of the universal property of the coproduct

$$\begin{tikzcd}
F(M) \arrow[dr] \arrow[drr, bend left, "F(\iota_M)"] \arrow[drrr, bend left] & & & \\
& F(M) \oplus F(N) \arrow[r] & F(M \oplus N) \arrow[r] & F(M) \oplus F(N) \\
F(N) \arrow[ur] \arrow[urr, bend right, "F(\iota_N)"'] \arrow[urrr, bend right] & & &
\end{tikzcd}$$

implies that $F(M) \oplus F(N) \to F(M \oplus N) \to F(M) \oplus F(N)$ is the identity map.

Now, we assume (b) and (c) and prove (a). Recall from the proof of the proposition 1.1.10 that the sum of two morphisms $\varphi_1, \varphi_2 : M \to N$ can be written as the composition

$$\begin{tikzcd}
M \arrow[r] \arrow[rrr, bend right, "\varphi_1+\varphi_2"'] & M \oplus M \arrow[r, "\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}"] & N \oplus N \arrow[r] & N.
\end{tikzcd}$$

We apply the functor F and consider the following diagram

$$\begin{tikzcd}
F(M) \arrow[r] \arrow[dr] & F(M \oplus M) \arrow[r, "F\left(\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}\right)"] & F(N \oplus N) \arrow[r] \arrow[d] & F(N), \\
& F(M) \oplus F(M) \arrow[u] \arrow[r, "\begin{pmatrix} F(\varphi_1) & 0 \\ 0 & F(\varphi_2) \end{pmatrix}"] & F(N) \oplus F(N), \arrow[ur] &
\end{tikzcd}$$

which we claim to be commutative. Observe that the composition of the morphisms on the top is $F(\varphi_1 + \varphi_2)$ and the composition of the morphisms on the bottom is $F(\varphi_1) + F(\varphi_2)$. The commutativity of the diagram then implies (a).

Both triangles commute by the very definition of the morphisms $F(M) \oplus F(M) \to F(M \oplus M)$ and $F(N \oplus N) \to F(N) \oplus F(N)$. The commutativity of the inner square is just as natural, but a little notationally awkward. Let's denote the morphisms involved as follows:

$$\begin{tikzcd}
F(M \oplus M) \arrow[r, "F(\psi)"] & F(N \oplus N) \arrow[d, "\beta"] \\
F(M) \oplus F(M) \arrow[u, "\alpha"] \arrow[r, "\widetilde{\psi}"] & F(N) \oplus F(N).
\end{tikzcd}$$

Recall that $\psi : M \oplus M \to N \oplus N$ is the unique morphism such that

$$\begin{aligned} \pi_1 \circ \psi \circ \iota_1 &= \varphi_1 & \qquad \pi_1 \circ \psi \circ \iota_2 &= 0 \\ \pi_2 \circ \psi \circ \iota_1 &= 0 & \qquad \pi_2 \circ \psi \circ \iota_2 &= \varphi_2. \end{aligned}$$

By applying the functor F to these relations and recalling that $F(\iota_i) = \alpha \circ \tilde{\iota_i}$ and $F(\pi_i) = \tilde{\pi_i} \circ \beta$, where $\tilde{\iota_i} : F(M) \to F(M) \oplus F(M)$ and $\tilde{\pi_i} : F(N) \oplus F(N) \to F(N)$ are the natural inclusions and projections, we get that $\beta \circ F(\psi) \circ \alpha$ satisfies the defining equations for

$$\begin{pmatrix} F(\varphi_1) & 0 \\ 0 & F(\varphi_2) \end{pmatrix}.$$

This proves the commutativity of the square. □

Our next goal is to prove that if C is a small category and A is an abelian category, then the category of all functors and natural transformations $\mathrm{Fun}(C, A)$ is also abelian. This is a generalization of a fact that will become very important to us in the future: the category of presheaves over an abelian category is abelian.

The reader might wonder the raison d'être *of the set-theoretic condition above. If* C *is not small, then the objects of* $\mathrm{Fun}(C, A)$ *doesn't even form a class. If it were a class, then a functor* $C \to A$ *would be a set, since a set is defined to be a collection that is a member of some class. But then we could use the axiom of replacement to deduce that the class of objects of* C *is a set.*

For that, we have to understand how some limits and colimits work in a functor category. The general statement is that "limits and colimits in a functor category are computed pointwise". We prefer to understand concretely the particular cases we're interested in, but the reader can find the general theorem in [3] (proposition 2.15.1) or in [33] (theorem 6.2.5).

We begin by a simple observation: the functor $C \to A$ which sends every object of C to the zero-object of A is a zero-object of $\mathrm{Fun}(C, A)$. Moreover, if F, G are objects of $\mathrm{Fun}(C, A)$, a natural transformation $F \to G$ is a zero-morphism if and only if all its components $F(C) \to G(C)$ are zero-morphisms in A.

Now, let's deal with kernels. Suppose that $\varphi : F \to G$ is a natural transformation in $\mathrm{Fun}(C, A)$. For each $C \in C$, the morphism $\varphi_C : F(C) \to G(C)$ has a kernel $\ker \varphi_C : K(C) \to F(C)$. We observe that this assignment is functorial. If $f : C \to D$ is a morphism in C, then the diagram

$$\begin{tikzcd} K(C) \arrow[r, "\ker \varphi_C"] & F(C) \arrow[r, "\varphi_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\ K(D) \arrow[r, "\ker \varphi_D"] & F(D) \arrow[r, "\varphi_D"] & G(D) \end{tikzcd}$$

is commutative and so the universal property of kernels will induce a morphism $K(C) \to K(D)$ making the diagram commute as long as the morphism

$$\begin{tikzcd}
K(C) \arrow[r, "\ker\varphi_C"] & F(C) \arrow[d, "F(f)"] & \\
& F(D) \arrow[r, "\varphi_D"] & G(D)
\end{tikzcd}$$

is zero. But this is evident since the commutativity of the diagram implies that this morphism is equal to

$$\begin{tikzcd}
K(C) \arrow[r, "\ker\varphi_C"] \arrow[rr, bend right, "0"'] & F(C) \arrow[r, "\varphi_C"] & G(C) \arrow[d, "G(f)"] \\
& & G(D).
\end{tikzcd}$$

Moreover, the uniqueness part of the universal property of kernels shows that if $G : D \to E$ is another morphism in C, then the bigger diagram

$$\begin{tikzcd}
K(C) \arrow[r, "\ker\varphi_C"] \arrow[d, dashed] \arrow[dd, dashed, bend right] & F(C) \arrow[r, "\varphi_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\
K(D) \arrow[r, "\ker\varphi_D"] \arrow[d, dashed] & F(D) \arrow[r, "\varphi_D"] \arrow[d, "F(g)"] & G(D) \arrow[d, "G(g)"] \\
K(E) \arrow[r, "\ker\varphi_E"] & F(E) \arrow[r, "\varphi_E"] & G(E)
\end{tikzcd}$$

commutes. We conclude that $C \mapsto K(C)$ defines a functor $\mathsf{C} \to \mathsf{A}$ and that $K \to F$ is a morphism in $\operatorname{Fun}(\mathsf{C}, \mathsf{A})$ whose composition with $\varphi : F \to G$ is zero. Does it satisfy the universal property of $\ker\varphi$? Let $\zeta : Z \to F$ be another natural transformation which satisfies $\varphi \circ \zeta = 0$. By the universal property of kernels, there exist unique morphisms $Z(C) \to K(C)$ for every object C of C making the diagram

$$\begin{tikzcd}
K(C) \arrow[r, "\ker\varphi_C"] & F(C) \arrow[r, shift left, "\varphi_C"] \arrow[r, shift right, "0"'] & G(C) \\
Z(C) \arrow[u, dashed] \arrow[ur, "\zeta_C"'] & &
\end{tikzcd}$$

commute. These morphisms form a natural transformation since, if $f : C \to D$ is a morphism in C, then the diagram

$$\begin{tikzcd}
K(C) \arrow[rr, "\ker \varphi_C"] \arrow[dr, "K(f)"] & & F(C) \arrow[dr, "F(f)"] & \\
& K(D) \arrow[rr, "\ker \varphi_D"] & & F(D) \\
Z(C) \arrow[uu, dashed] \arrow[uurr, "\zeta_C"] \arrow[dr, "Z(f)"'] & & & \\
& Z(D) \arrow[uu, dashed] \arrow[uurr, "\zeta_D"] & &
\end{tikzcd}$$

commutes since $\ker \varphi_D$ is a monomorphism. This proves that $K \to F$ satisfies the universal property of $\ker \varphi$. Since $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is a preadditive category (with the addition of morphisms given pointwise), this implies that a morphism $\varphi : F \to G$ in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is a monomorphism if and only if $\ker \varphi$ is the zero-morphism $0 \to F$ and if and only if it is a monomorphism pointwise.

It should be clear that the same argument shows that cokernels in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ exist and are computed pointwise. Moreover, a morphism $\varphi : F \to G$ in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is an epimorphism if and only if $\operatorname{coker} \varphi$ is the zero-morphism $G \to 0$ and if and only if it is an epimorphism pointwise.

Finally, basically the same arguments show that, if F and G are two objects of $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$, the functor $F \oplus G$ defined by

$$(F \oplus G)(C) := F(C) \oplus G(C) \qquad \text{and} \qquad (F \oplus G)(f) := \begin{pmatrix} F(f) & 0 \\ 0 & G(f) \end{pmatrix}$$

satisfies the universal property of products and coproducts in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$, with the natural injections and projections being given by the respective pointwise injections and projections.

We're now ready to prove our desired result.

Proposition 1.5.2 Let C be a small category and A be an abelian category. Then the category of all functors and natural transformations $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is abelian.

Proof. After all our preliminary work, all there's left to prove is that every monomorphism is the kernel of its cokernel and that every epimorphism is the cokernel of its kernel. This also follows quickly from our previous discussion: if $\varphi : F \to G$ is a monomorphism then its components $\varphi_C : F(C) \to G(C)$, for every object C of C, are monic. Since A is abelian, each φ_C is the kernel of its cokernel. But kernels and cokernels are computed pointwise and so φ is also the kernel of its cokernel. The same argument shows that every epimorphism is the cokernel of its kernel. □

We illustrate how to apply the corollary 1.2.2 by proving that the category of additive functors is also abelian.

> **Corollary 1.5.3** Let C be a small additive category and A be an abelian category. Then the full subcategory $\mathrm{Add}(\mathsf{C}, \mathsf{A})$ of $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$, composed of additive functors and natural transformations, is abelian.

Proof. It is clear that the zero-object of $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$ is additive, and so it is also the zero-object of $\mathrm{Add}(\mathsf{C}, \mathsf{A})$. If F and G are two additive functors, their direct sum acts on morphisms by

$$(F \oplus G)(f) = \begin{pmatrix} F(f) & 0 \\ 0 & G(f) \end{pmatrix}.$$

Since the sum of morphisms is represented by the sum of matrices, the additivity of both F and G implies that of $F \oplus G$. Finally, we show that, if $\varphi : F \to G$ is a morphism in $\mathrm{Add}(\mathsf{C}, \mathsf{A})$ and $\ker \varphi : K \to F$ is its kernel in $\mathrm{Fun}(\mathsf{C}, \mathsf{A})$, K is an additive functor. Indeed, if $f, g : C \to D$ are two morphisms in C,

$$\begin{aligned} \ker \varphi_D \circ K(f+g) &= F(f+g) \circ \ker \varphi_C = (F(f) + F(g)) \circ \ker \varphi_C \\ &= F(f) \circ \ker \varphi_C + F(g) \circ \ker \varphi_C \\ &= \ker \varphi_D \circ K(f) + \ker \varphi_D \circ K(g) = \ker \varphi_D \circ (K(f) + K(g)), \end{aligned}$$

and so $K(f+g) = K(f) + K(g)$ by the fact that $\ker \varphi_D$ is a monomorphism. The same argument shows that the target of $\operatorname{coker} \varphi$ is also additive. □

We'll now delve into the relationship between functors and exact sequences. Unfortunately, being additive does *not* guarantee that a functor preserves exact sequences.[5] For example, consider the exact sequence of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r] & \frac{\mathbb{Z}}{2\mathbb{Z}} \arrow[r] & 0, \end{tikzcd}$$

where the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by two. Upon tensorization by $\mathbb{Z}/2\mathbb{Z}$ we get the sequence

$$\begin{tikzcd} 0 \arrow[r] & \frac{\mathbb{Z}}{2\mathbb{Z}} \arrow[r, "0"] & \frac{\mathbb{Z}}{2\mathbb{Z}} \arrow[r] & \frac{\mathbb{Z}}{2\mathbb{Z}} \arrow[r] & 0, \end{tikzcd}$$

which is not exact since the zero-morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is not a monomorphism. The additive functors that indeed preserve some kind of exact sequences are so special that they deserve a name.

---

[5] Or perhaps that's a blessing, for this issue is at the heart of homological algebra.

Definition 1.5.2 — Exact functor. Let $F : A \to B$ be an additive functor between abelian categories. Then F is said to be *left exact* when it preserves exact sequences of the form

$$0 \longrightarrow M \longrightarrow N \longrightarrow P,$$

right exact when it preserves exact sequences of the form

$$M \longrightarrow N \longrightarrow P \longrightarrow 0,$$

and *exact* when it preserves short exact sequences.

We observe that our discussion right after the definition 1.4.2 implies that an exact functor preserves exact sequences of any length, not only short exact sequences.

Proposition 1.5.4 Let $F : A \to B$ be an additive functor between abelian categories. The following equivalences hold:

(a) F is left exact if and only if it preserves finite limits;

(b) F is right exact if and only if it preserves finite colimits;

(c) F is exact if and only if it preserves finite limits and finite colimits.

Proof. By duality, it suffices to prove (a). We observe that a sequence of the form

$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

is exact if and only if $\varphi = \ker\psi$. This implies right away that if F preserves finite limits, then it preserves kernels and so it is left exact. For the converse, recall that finite limits can be built up from binary products, terminal objects and equalizers. (Proposition 2.8.2 in [3].) Since F is additive, it preserves binary products and zero-objects. Moreover, if F is left exact, then it preserves kernels. It suffices then to show that F preserves equalizers. But the equalizer of a pair $\varphi, \psi : M \to N$ is simply the kernel of $\varphi - \psi$. The result follows. ☐

More often than not, what we'll use to prove that a functor is left or right exact is the corollary below, which follows from our good old mottos "right adjoints preserve limits" and its dual "left adjoints preserve colimits".[6]

Corollary 1.5.5 Let $F : A \to B$ be an additive functor between abelian categories. If F is a right adjoint then it is left exact and if F is a left adjoint then it is right exact.

---

[6] We remember that right adjoints preserve limits by the mnemonic *RAPL*.

1.6 Diagram chasing

In the abelian category A-Mod of modules over a ring A, exact sequences have simple characterizations in terms of elements. Indeed, the sequence of A-modules

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

is exact if and only if $\psi(\varphi(m)) = 0$ for all $m \in M$ and if $\psi(n) = 0$, for some $n \in N$, implies the existence of $m \in M$ such that $n = \varphi(m)$. Using this, proofs involving exact sequences can usually be done by pointing fingers to a diagram and observing the fate of some elements. This technique is called *diagram chasing*.

To illustrate this technique, we prove the following result in two ways; first using universal properties and then, in A-Mod, using diagram chasing.

Proposition 1.6.1 — Four lemma. Consider the following diagram with exact rows in an abelian category A:

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 .
\end{tikzcd}$$

If β and δ are monomorphisms and α is an epimorphism, then γ is a monomorphism. Dually, if α and γ are epimorphisms and δ is a monomorphism, then β is an epimorphism.

As usual, we prove only the first part of the result, since the second part follows by duality.

Proof using universal properties. Let $\rho : P \to M_3$ be a morphism such that $\gamma \circ \rho = 0$. Our goal is to prove that $\rho = 0$. Since the diagram commutes, the morphism

$$\begin{tikzcd}
 & & P \arrow[d, "\rho"] & \\
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4
\end{tikzcd}$$

is zero and, as δ is monic, so is $P \to M_3 \to M_4$. The universal property of kernels then implies that ρ factors through the kernel $K \to M_3$ of $M_3 \to M_4$, which coincides with the image of $M_2 \to M_3$ by exactness.

$$\begin{tikzcd}
& & & P \arrow[dl, dashed] \arrow[d, "\rho"] & \\
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

We consider the pullback $M_2 \times_K P$ and observe that the commutativity of the diagram implies that the morphism below is zero.

$$\begin{tikzcd}
& M_2 \times_K P \arrow[rr] \arrow[d] & & P \arrow[dl] \arrow[d, "\rho"] & \\
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

By the universal property of kernels, $M_2 \times_K P \to M_2 \to N_2$ factors through the kernel $K' \to N_2$ of $N_2 \to N_3$, which coincides with the image of $N_1 \to N_2$ by exactness.

$$\begin{tikzcd}
& & M_2 \times_K P \arrow[rr] \arrow[d] \arrow[ddl, dashed] & & P \arrow[dl] \arrow[d, "\rho"] & \\
M_1 \arrow[rr] \arrow[d, "\alpha"] & & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & K' \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

We consider the pullback $M_1 \times_{K'} (M_2 \times_K P)$ and observe that, since β is a monomorphism, the upper square (in black below) commutes.

$$\begin{tikzcd}
M_1 \times_{K'} (M_2 \times_K P) \arrow[rr] \arrow[d] & & M_2 \times_K P \arrow[rr] \arrow[d] \arrow[ddl] & & P \arrow[dl] \arrow[d, "\rho"] & \\
M_1 \arrow[rr] \arrow[d, "\alpha"] & & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & K' \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

Remark that both $M_2 \to K$ and $M_1 \to N_1 \to K'$ are epimorphisms. The corollary 1.3.4 then implies that so are the arrows in black below.

$$\begin{tikzcd}
M_1 \times_{K'} (M_2 \times_K P) \arrow[rr] \arrow[d] & & M_2 \times_K P \arrow[rr] \arrow[d] \arrow[ddl] & & P \arrow[d, "\rho"] \arrow[dl] & \\
M_1 \arrow[rr] \arrow[d, "\alpha"] & & M_2 \arrow[r] \arrow[d, "\beta"] & K \arrow[r] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & K' \arrow[r] & N_2 \arrow[rr] & & N_3 \arrow[r] & N_4
\end{tikzcd}$$

The composition of the arrows above with ρ is zero, since $M_1 \to M_2 \to M_3$ is. But the fact that they are epic implies that $\rho = 0$, finishing the proof. □

We now prove the same result, when $\mathsf{A} = A\text{-}\mathsf{Mod}$, using diagram chase. Observe that, since we're now proving this result for only one abelian category, we can't use a duality argument. (The opposite category of $A\text{-}\mathsf{Mod}$ is rarely a category of modules.) Nevertheless, we'll still only prove the first part below, for our last proof took care of both parts. We encourage the reader to remark that, in the proof below, every step is the only one possible.

Proof by diagram chasing. Let m be an element of M_3 such that $\gamma(m) = 0$. Our goal is to prove that $m = 0$. Observe that m is sent to 0 in N_4 by the composition

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & m \arrow[r] \arrow[d, mapsto, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & 0 \arrow[r, mapsto] & 0.
\end{tikzcd}$$

Since the diagram commutes, m is also sent to 0 by going through the other side of the square

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & m \arrow[r, mapsto] \arrow[d, "\gamma"] & ? \arrow[d, mapsto, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0.
\end{tikzcd}$$

But δ is injective, so m is in the kernel of the morphism $M_3 \to M_4$. (That is, our "?" above is actually zero.) By exactness of the top row, there exists $m' \in M_2$, which is sent to m by $M_2 \to M_3$.

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & m' \arrow[r, mapsto] \arrow[d, "\beta"] & m \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Since the middle square commutes, $n := \beta(m')$ is in the kernel of $N_2 \to N_3$. So, by exactness of the lower row, there exists $n' \in N_1$ whose image through $N_1 \to N_2$ is n.

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & m' \arrow[r] \arrow[d, mapsto, "\beta"] & m \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[d, "\delta"] \\
n' \arrow[r, mapsto] & n \arrow[r, mapsto] & 0 \arrow[r] & 0
\end{tikzcd}$$

The morphism α is epic, so there exists $m'' \in M_1$ which is sent to n'. This element is actually sent to m' via $M_1 \to M_2$ due to the fact that β is monic. We conclude that m is the image of m'' under the composition $M_1 \to M_2 \to M_3$.

$$\begin{tikzcd}
m'' \arrow[r, mapsto] \arrow[d, "\alpha"] & m' \arrow[r, mapsto] \arrow[d, "\beta"] & m \arrow[r] \arrow[d, "\gamma"] & 0 \arrow[d, "\delta"] \\
n' \arrow[r] & n \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

But this composition is zero, proving the result. ☐

By gluing both versions of the four lemma, we obtain the corollary below.

Corollary 1.6.2 — Five lemma. Consider the following diagram with exact rows in an abelian category A:

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[r] \arrow[d, "\delta"] & M_5 \arrow[d, "\varepsilon"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5.
\end{tikzcd}$$

If β and δ are isomorphisms, α is an epimorphism, and ε is a monomorphism, then γ is an isomorphism.

Proof. The first part of the four lemma, applied to the diagram

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4,
\end{tikzcd}$$

yields that γ is a monomorphism. Similarly, the second part of the four lemma, applied to the diagram

$$\begin{tikzcd}
M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[r] \arrow[d, "\delta"] & M_5 \arrow[d, "\varepsilon"] \\
N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5,
\end{tikzcd}$$

yields that γ is an epimorphism. This concludes the proof. $\square$

The preceding discussion hopefully conveyed that proofs by diagram chasing are often simpler than their arrow-theoretic counterparts. It would be great if we could use the same technique even when dealing with abelian categories other than $A\text{-}\mathsf{Mod}$. The theorem below establishes precisely that.

Theorem 1.6.3 — Freyd-Mitchell. Let A be a small abelian category. Then there exists a fully faithful exact embedding of A into $A\text{-}\mathsf{Mod}$ for some (not necessarily commutative) ring A.

While all the necessary prerequisites for the (unfortunately long) proof of this result were already discussed, we prefer to direct the interested reader to the wonderful proof in [4] and confine ourselves to an explanation of how this result is used in practice.

Let $V : \mathsf{A} \to A\text{-}\mathsf{Mod}$ be the functor given by the Freyd-Mitchell theorem. For now, we define a *pseudo-element* m of an object $M \in \mathsf{A}$ to be an element of $V(M)$. We shall abuse notation and write $m \in M$ for this relation. The action of a morphism $\varphi : M \to N$, denoted as $\varphi(m)$, on a pseudo-element m is given simply by $V(\varphi)(m)$. We gather a few properties of those notions.

Proposition 1.6.4 Let A be a small abelian category. If $\varphi : M \to N$ is a morphism in A, we have that:

(a) φ is monic if and only if for all $m \in M$, $\varphi(m) = 0$ implies $m = 0$;

(b) φ is epic if and only if for all $n \in N$, there exists $m \in M$ such that $\varphi(m) = n$;

(c) we may construct a morphism φ by describing its action of pseudo-elements.

Moreover,

(d) two morphisms $\varphi_1, \varphi_2 : M \to N$ are equal if and only if $\varphi_1(m) = \varphi_2(m)$ for all $m \in M$;

(e) a sequence $M \xrightarrow{\varphi} N \xrightarrow{\psi} N$ is exact if and only if $\psi(\varphi(m)) = 0$ for all $m \in M$ and if $\psi(n) = 0$, for some $n \in N$, implies the existence of $m \in M$ such that $n = \varphi(m)$.

Proof. The item (c) translates the fullness of the functor V in theorem 1.6.3, and the item (d) translates it's faithfulness. Since V is exact, it preserves finite limits and colimits; this gives one direction on the items (a), (b) and (e). The other direction follows from the fact that a fully faithful functor reflects limits and colimits, which is clear from their universal properties. $\square$

Finally, we address the elephant in the room: most abelian categories are not small.

This is not as bad as it seems, and we explain why. Let A be an abelian category and D be a diagram in A. Consider the sequence

$$\mathsf{B}_0 \subset \mathsf{B}_1 \subset \cdots \subset \mathsf{B}_n \subset \cdots ,$$

where B_0 is the full subcategory of A generated by D and B_{n+1} is the full subcategory of A generated by the limits and colimits of all finite diagrams in B_n. Then

$$\mathsf{B} := \bigcup_{n=0}^{\infty} \mathsf{B}_n$$

is a full subcategory of A stable under finite limits and colimits. In particular, B is an abelian category due to the corollary 1.2.2. If the diagram D is small (which is the case in basically all applications), so is the abelian category B, and then we can apply the theorem 1.6.3 in B.

In a nutshell, the Freyd-Mitchell theorem allows us prove basically every result about exact sequences in abelian categories as if we were in a category of modules. And we may even use duality arguments!

Henceforth, we'll prefer arrow-theoretic constructions whenever they aren't too troublesome, but we will freely use elements when they simplify or shed light on some arguments.

We end this section with arguably the most important diagram chase: the *snake lemma*. Its statement involves a diagram of the form

$$\begin{tikzcd}
 & M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3,
\end{tikzcd}$$

whose rows are exact, where we expand the kernels and cokernels of the vertical morphisms and insert the natural morphisms induced from the universal properties:

$$\begin{tikzcd}
 & K_\alpha \arrow[r, dashed] \arrow[d, "\ker\alpha"] & K_\beta \arrow[r, dashed] \arrow[d, "\ker\beta"] & K_\gamma \arrow[d, "\ker\gamma"] & \\
 & M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[d, "\operatorname{coker}\alpha"] & N_2 \arrow[r] \arrow[d, "\operatorname{coker}\beta"] & N_3 \arrow[d, "\operatorname{coker}\gamma"] & \\
 & C_\alpha \arrow[r, dashed] & C_\beta \arrow[r, dashed] & C_\gamma. &
\end{tikzcd}$$

We're now in a position to state this important result.

Theorem 1.6.5 — Snake lemma. Consider the following commutative diagram with exact rows in an abelian category:

$$\begin{tikzcd}
& M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3.
\end{tikzcd}$$

We denote by K_α, K_β and K_γ the sources of $\ker\alpha$, $\ker\beta$ and $\ker\gamma$. Similarly, C_α, C_β and C_γ denote the targets of the cokernels thereof. Then, there exists a morphism $\delta : K_\gamma \to C_\alpha$ making the sequence

$$\begin{tikzcd}
K_\alpha \arrow[r] & K_\beta \arrow[r] & K_\gamma \arrow[r, "\delta"] & C_\alpha \arrow[r] & C_\beta \arrow[r] & C_\gamma
\end{tikzcd}$$

exact. Moreover, if $M_1 \to M_2$ is a monomorphism, then so is $K_\alpha \to K_\beta$, and if $N_2 \to N_3$ is an epimorphism, then so is $C_\beta \to C_\gamma$.

Before we delve into the proof, we observe that, even though there may be many morphisms $\delta : K_\gamma \to C_\alpha$ which satisfy the conclusion above[7], there's a canonical one that will be the one in consideration whenever we talk about the snake lemma.

We construct the morphism δ using elements as follows: let m be an element of K_γ. Since $\ker\gamma$ is a monomorphism, we can view m naturally as an element of M_3. Due to the fact that $M_2 \to M_3$ is an epimorphism, there exists a lift m' of m to M_2, which we then map to N_2 as $\beta(m')$. By the commutativity of the diagram, the image of $\beta(m')$ to N_3 is zero, proving that $\beta(m')$ is in the image of $N_1 \to N_2$. Since the latter is monic, we denote the element of N_1 whose image by $N_1 \to N_2$ is $\beta(m')$ by the same symbol. Finally, $\delta(m)$ is the image of $\beta(m')$ in the cokernel of α.

$$\begin{tikzcd}
& K_\alpha \arrow[r] \arrow[d, "\ker\alpha"] & K_\beta \arrow[r] \arrow[d, "\ker\beta"] & m \arrow[d, mapsto, "\ker\gamma"] \\
& M_1 \arrow[r] \arrow[d, "\alpha"] & m' \arrow[r, mapsto] \arrow[d, mapsto, "\beta"] & m \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \beta(m') \arrow[r, mapsto] \arrow[d, mapsto, "\operatorname{coker}\alpha"] & \beta(m') \arrow[r] \arrow[d, "\operatorname{coker}\beta"] & N_3 \arrow[d, "\operatorname{coker}\gamma"] \\
& \delta(m) \arrow[r] & C_\beta \arrow[r] & C_\gamma
\end{tikzcd}$$

In order for this morphism to be well-defined, we need to check whether a different choice for the lift m' would change the image $\delta(m)$. If m'' is another choice, then

---

[7] If δ satisfies the conclusion of the snake lemma, then so does $-\delta$.

$\mathfrak{m}' - \mathfrak{m}''$ is in the kernel of $M_2 \to M_3$ and so in the image of $M_1 \to M_2$. Let $\tilde{\mathfrak{m}} \in M_1$ be one element mapping to $\mathfrak{m}' - \mathfrak{m}''$. Its image in C_α is zero, since $M_1 \to N_1 \to C_\alpha$ is the zero-morphism.

$$\begin{tikzcd}
 & \tilde{\mathfrak{m}} \arrow[r, mapsto] \arrow[d, mapsto, "\alpha"] & \mathfrak{m}' - \mathfrak{m}'' \arrow[r, mapsto] \arrow[d, "\beta"] & 0 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \alpha(\tilde{\mathfrak{m}}) \arrow[r] \arrow[d, mapsto, "\operatorname{coker} \alpha"] & N_2 \arrow[r] \arrow[d, "\operatorname{coker} \beta"] & N_3 \arrow[d, "\operatorname{coker} \gamma"] & \\
 & 0 \arrow[r] & C_\beta \arrow[r] & C_\gamma &
\end{tikzcd}$$

The commutativity of the diagram then implies that $\delta(\mathfrak{m})$ is independent of the choice of the lift.

Proof of theorem 1.6.5. ☐

2 Complexes and cohomology

In the previous chapter, we saw that only the most distinguished additive functors turns out to be exact. Nevertheless, the image of an exact sequence

$$M \xrightarrow{\varphi} N \xrightarrow{\psi} P$$

by an additive functor F is still special, for it satisfies $F(\psi) \circ F(\varphi) = F(\psi \circ \varphi) = F(0) = 0$. The sequences of objects and morphisms in which the composition of two consecutive morphisms is zero are called *complexes* and compose the main topic of the present chapter. We'll see that there are many contexts in which associating a particular complex to a mathematical object provides useful information about the aforesaid object.

2.1 Basic definitions

We begin with the precise definition of a complex.

Definition 2.1.1 — Complex. Let A be a (not necessarily abelian) category. A *cochain complex* $(M^\bullet, d^\bullet)$ in A is a sequence of objects and morphisms

$$\cdots \xrightarrow{d^{i-2}} M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^{i+1}} \cdots$$

such that $d^i \circ d^{i-1} = 0$ for all i.

In some applications, it is useful for the indices to be descending. In this case, the indices are usually written as subscripts

$$\cdots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \cdots$$

and the corresponding object is said to be a *chain complex*. Since most of the complexes that we'll encounter are *co*chain complexes, we'll just call them *complexes* and denote them by $M^\bullet$. Of course, we can always set $M^i := M_{-i}$ and see a chain complex $M_\bullet$ as the cochain complex $M^{-\bullet}$.

Also important are the ways complexes can interact with each other. For that, we gather all the complexes in A in a new category $\mathsf{C}(\mathsf{A})$.

Definition 2.1.2 — Category of complexes. Let A be a category. An object in the *category of complexes* $C(A)$ is a complex in A and a morphism $\psi^\bullet : M^\bullet \to N^\bullet$ is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^{i-1} \arrow[r, "d^{i-1}_{M^\bullet}"] \arrow[d, "\psi^{i-1}"] & M^i \arrow[r, "d^i_{M^\bullet}"] \arrow[d, "\psi^i"] & M^{i+1} \arrow[r] \arrow[d, "\psi^{i+1}"] & \cdots \\
\cdots \arrow[r] & N^{i-1} \arrow[r, "d^{i-1}_{N^\bullet}"] & N^i \arrow[r, "d^i_{N^\bullet}"] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

in A. The morphisms $d^i : M^i \to M^{i+1}$ are said to be the *differentials* of the complex.

Other usual variants of the category $C(A)$ may be concocted by considering complexes which are bounded in some sense. For example, we let $C^+(A)$ denote the full subcategory of $C(A)$ composed of the complexes $M^\bullet$ which are bounded below, i.e., for which $M^i = 0$ for all $i \ll 0$. Similarly, we consider the categories $C^-(A)$ of bounded-above complexes and $C^b(A)$ of complexes which are bounded above and below. A shorthand notation for all these categories is $C^*(A)$.

Proposition 2.1.1 Let A be an abelian category. Then the categories of complexes $C^*(A)$ are abelian.

Proof. Due to the corollary 1.2.2, it suffices to prove that $C(A)$ is abelian. Consider the category Z, which has an object for each integer and a single non-trivial morphism between each consecutive integers (from the smallest to the biggest). Then $C(A)$ is a full subcategory of $\mathrm{Fun}(Z, A)$, which is abelian by the proposition 1.5.2. Appealing once again to the corollary 1.2.2, it suffices to see that the category of complexes is closed under direct sums, kernels and cokernels.

Binary direct sums of complexes form another complex since, for all i, the diagram

$$\begin{tikzcd}
M^{i-1} \arrow[r, "d^{i-1}_{M^\bullet}"] & M^i \arrow[r, "d^i_{M^\bullet}"] & M^{i+1} \arrow[d, "\iota^{i+1}_{M^\bullet}"] \\
M^{i-1} \oplus N^{i-1} \arrow[r] \arrow[u, "\pi^{i-1}_{M^\bullet}"] & M^i \oplus N^i \arrow[r] & M^{i+1} \oplus N^{i+1}
\end{tikzcd}$$

commutes. Moreover, if $\psi^\bullet : M^\bullet \to N^\bullet$ is a morphism of complexes, we have a commutative diagram

$$\begin{tikzcd}
K^{i-1} \arrow[r] \arrow[d, "\ker \psi^{i-1}"] & K^i \arrow[r] \arrow[d, "\ker \psi^i"] & K^{i+1} \arrow[d, "\ker \psi^{i+1}"] \\
M^{i-1} \arrow[r, "d^{i-1}_{M^\bullet}"] & M^i \arrow[r, "d^i_{M^\bullet}"] & M^{i+1}
\end{tikzcd}$$

By the complex condition, $K^{i-1} \to K^i \to K^{i+1} \to M^{i+1}$ is the zero-morphism and, since $\ker \psi^{i+1}$ is a monomorphism, $K^{i-1} \to K^i \to K^{i+1}$ is also already zero, proving that the category of complexes is closed under kernels. A dual argument shows that it is closed under cokernels. □

We now observe some natural functors which involve the category of complexes. First of all, as it was seen in the proof above, our category A can be embedded in $\mathsf{C}^*(\mathsf{A})$. Indeed, the functor $\iota : \mathsf{A} \to \mathsf{C}^*(\mathsf{A})$ which sends an object A of A to the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \underbrace{A}_{\text{degree } 0} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is fully faithful (it is also exact when A is abelian). Another natural functor on $\mathsf{C}^*(\mathsf{A})$ is the shift functor:

$$\begin{aligned} \mathsf{C}^*(\mathsf{A}) &\to \mathsf{C}^*(\mathsf{A}) \\ M^\bullet &\mapsto M[n]^\bullet, \end{aligned}$$

defined by $M[n]^i := M^{n+i}$ and $d^i_{M[n]^\bullet} := (-1)^n d^{n+i}_{M^\bullet}$. The sign on the differential doesn't change the isomorphism class of the complex but simplifies some other equations. Also, an additive functor between additive categories $F : \mathsf{A} \to \mathsf{B}$ determines a functor between the categories of complexes

$$\mathsf{C}^*(F) : \mathsf{C}^*(\mathsf{A}) \to \mathsf{C}^*(\mathsf{B})$$

given by setting the image of $M^\bullet$ to be the complex defined by $F(M^i)$ and $F(d^i_{M^\bullet})$. Whenever there's no risk of confusion, we'll denote this functor simply by F.

There's another, even more interesting, functor defined on the category of complexes $\mathsf{C}^*(\mathsf{A})$ when A is abelian. Consider a complex $M^\bullet$. The complex condition $d^i \circ d^{i-1} = 0$ and the universal property of kernels imply that d^{i-1} factors through $\ker d^i$:

$$\begin{tikzcd}
\cdots \arrow[r, "d^{i-2}"] & M^{i-1} \arrow[rr, "d^{i-1}"] \arrow[dr, dashed] & & M^i \arrow[r, "d^i"] & M^{i+1} \arrow[r, "d^{i+1}"] & \cdots . \\
& & K^i \arrow[ur, "\ker d^i"'] & & &
\end{tikzcd}$$

But $\ker d^i$ is a monomorphism and so the universal property of images yields a unique factorization of $\operatorname{im} d^{i-1}$ through $\ker d^i$:

$$\begin{tikzcd}
& & I^{i-1} \arrow[dr, "\operatorname{im} d^{i-1}"] \arrow[dd, dashed] & & & \\
\cdots \arrow[r, "d^{i-2}"] & M^{i-1} \arrow[rr] \arrow[ur] \arrow[dr] & & M^i \arrow[r, "d^i"] & M^{i+1} \arrow[r, "d^{i+1}"] & \cdots . \\
& & K^i \arrow[ur, "\ker d^i"'] & & &
\end{tikzcd}$$

The induced morphism $I^{i-1} \to K^i$ is always a monomorphism (for $\operatorname{im} d^{i-1}$ is) and is epic if and only if the complex is exact at M^i. Thus, its cokernel measures the lack of exactness of the complex at M^i.

Definition 2.1.3 — Cohomology. Let $M^\bullet$ be a complex in an abelian category A. Its i-*th cohomology*, denoted $H^i(M^\bullet)$, is the target of the cokernel of the induced morphism $I^{i-1} \to K^i$ as above.

We affirm that the assignment $M^\bullet \mapsto H^i(M^\bullet)$ defines an additive functor $\mathsf{C}^*(\mathsf{A}) \to \mathsf{A}$. Indeed, let $\psi^\bullet : M^\bullet \to N^\bullet$ be a morphism of complexes. By the universal property of kernels and cokernels, we have induced morphisms

$$\begin{tikzcd}
K^i_{M^\bullet} \arrow[r, "\ker d^i_{M^\bullet}"] \arrow[d, dashed] & M^i \arrow[r, "d^i_{M^\bullet}"] \arrow[d, "\psi^i"] & M^{i+1} \arrow[r, "\operatorname{coker} d^i_{M^\bullet}"] \arrow[d, "\psi^{i+1}"] & C^i_{M^\bullet} \arrow[d, dashed] \\
K^i_{N^\bullet} \arrow[r, "\ker d^i_{N^\bullet}"] & N^i \arrow[r, "d^i_{N^\bullet}"] & N^{i+1} \arrow[r, "\operatorname{coker} d^i_{N^\bullet}"] & C^i_{N^\bullet}.
\end{tikzcd}$$

In order for the universal property of cokernels to induce a morphism $H^i(\psi^\bullet) : H^i(M^\bullet) \to H^i(N^\bullet)$ making the diagram

$$\begin{tikzcd}
 & & I^{i-1}_{M^\bullet} \arrow[dl] \arrow[dr, "\operatorname{im} d^{i-1}_{M^\bullet}"] & & \\
H^i(M^\bullet) \arrow[d, dashed, "H^i(\psi^\bullet)"] & K^i_{M^\bullet} \arrow[l] \arrow[rr, "\ker d^i_{M^\bullet}"] \arrow[d] & & M^i \arrow[r, "d^i_{M^\bullet}"] \arrow[d, "\psi^i"] & M^{i+1} \arrow[d, "\psi^{i+1}"] \\
H^i(N^\bullet) & K^i_{N^\bullet} \arrow[l] \arrow[rr, "\ker d^i_{N^\bullet}"] & & N^i \arrow[r, "d^i_{N^\bullet}"] & N^{i+1} \\
 & & I^{i-1}_{N^\bullet} \arrow[ul] \arrow[ur, "\operatorname{im} d^{i-1}_{N^\bullet}"'] & &
\end{tikzcd}$$

commute, we have to show that the morphism $I^{i-1}_{M^\bullet} \to K^i_{M^\bullet} \to K^i_{N^\bullet} \to H^i(N^\bullet)$ is zero. Since $I^{i-1}_{N^\bullet} \to K^i_{N^\bullet} \to H^i(N^\bullet)$ is the zero-morphism, it suffices to construct a morphism $I^{i-1}_{M^\bullet} \to I^{i-1}_{N^\bullet}$ which factors $I^{i-1}_{M^\bullet} \to K^i_{M^\bullet} \to K^i_{N^\bullet}$. This morphism is induced by the universal property of kernels using the fact that $\operatorname{im} = \ker(\operatorname{coker})$:

$$\begin{tikzcd}
K^{i-1}_{M^\bullet} \arrow[r, "\ker d^{i-1}_{M^\bullet}"] \arrow[d] & M^{i-1} \arrow[r, "\operatorname{coim} d^{i-1}_{M^\bullet}"] \arrow[d, "\psi^{i-1}"] & I^{i-1}_{M^\bullet} \arrow[r, "\operatorname{im} d^{i-1}_{M^\bullet}"] \arrow[d, dashed] & M^i \arrow[r, "\operatorname{coker} d^{i-1}_{M^\bullet}"] \arrow[d, "\psi^i"] & C^{i-1}_{M^\bullet} \arrow[d] \\
K^{i-1}_{N^\bullet} \arrow[r, "\ker d^{i-1}_{N^\bullet}"] & N^{i-1} \arrow[r, "\operatorname{coim} d^{i-1}_{N^\bullet}"] & I^{i-1}_{N^\bullet} \arrow[r, "\operatorname{im} d^{i-1}_{N^\bullet}"] & N^i \arrow[r, "\operatorname{coker} d^{i-1}_{M^\bullet}"] & C^{i-1}_{N^\bullet}.
\end{tikzcd}$$

The left-hand side of the diagram commutes due to the universal property of cokernels and the fact that $\operatorname{coim} = \operatorname{coker}(\ker)$. The uniqueness of the induced morphism on cohomology implies right-away that H^i preserves the composition of morphisms and that it is additive.

In A-Mod, the i-th cohomology of a complex is simply given by $\ker d^i / \operatorname{im} d^{i-1}$ and, for a morphism $\psi^\bullet : M^\bullet \to N^\bullet$ of complexes, the induced morphism on cohomology is nothing but

$$H^i(\psi^\bullet) : [m] \mapsto [\psi^i(m)].$$

As it will become clear in the next sections, the morphisms of complexes $\psi^\bullet : M^\bullet \to N^\bullet$ which induce an isomorphism in cohomology are important and deserve a name.

Definition 2.1.4 — Quasi-isomorphism. A morphism of complexes $\psi^\bullet : M^\bullet \to N^\bullet$ is said to be a *quasi-isomorphism* if, for all i, the induced morphism $H^i(\psi^\bullet) : H^i(M^\bullet) \to H^i(N^\bullet)$ is an isomorphism.

We observe that we can also see cohomology as a functor $\mathsf{C}^*(\mathsf{A}) \to \mathsf{C}^*(\mathsf{A})$, where the image of a complex $M^\bullet$ is a complex $H^\bullet(M^\bullet)$ which has $H^i(M^\bullet)$ as objects and zero-morphisms as differentials.

An important property of cohomology is that it commutes with exact functors.

Proposition 2.1.2 Let $F : \mathsf{A} \to \mathsf{B}$ be an exact functor between abelian categories and $M^\bullet$ a complex in A. Then $H^\bullet(F(M^\bullet)) = F(H^\bullet(M^\bullet))$.

Proof. We construct the i-th cohomology group of $F(M^\bullet)$. Since F is additive,

$$\begin{tikzcd}
\cdots \arrow[r] & F(M^{i-1}) \arrow[r, "F(d^{i-1})"] & F(M^i) \arrow[r, "F(d^i)"] & F(M^{i+1}) \arrow[r] & \cdots
\end{tikzcd}$$

is indeed a complex. Due to the fact that F preserves finite limits and finite colimits, $F(\operatorname{im} d^{i-1})$ is the image of $F(d^{i-1})$ and $F(\ker d^i)$ is the kernel of $F(d^i)$.

$$\begin{tikzcd}
& & F(I^{i-1}) \arrow[dr, "F(\operatorname{im} d^{i-1})"] \arrow[dd, dashed] & & & \\
\cdots \arrow[r] & F(M^{i-1}) \arrow[ur] \arrow[dr] \arrow[rr] & & F(M^i) \arrow[r, "F(d^i)"] & F(M^{i+1}) \arrow[r] & \cdots \\
& & F(K^i) \arrow[ur, "F(\ker d^i)"'] & & &
\end{tikzcd}$$

Moreover, by the uniqueness in the universal property of images, the induced morphism $F(I^{i-1}) \to F(K^i)$ coincides with the image of the induced morphism $I^{i-1} \to K^i$ by F. Then, since F preserves finite colimits, the cokernel of $F(I^{i-1}) \to F(K^i)$ is simply the image of the cokernel of $I^{i-1} \to K^i$ by F, proving that $H^i(F(M^\bullet)) = F(H^i(M^\bullet))$. □

Since direct sums are limits and colimits at the same time, the direct sum functor preserves finite limits and finite colimits. The proposition 1.5.4 then implies that it is exact. As additive functors are precisely those that preserve finite direct sums, the preceding proposition gives another proof that cohomology defines an additive functor.

Before we move on, we observe that our definition of the cohomology of a complex is somewhat asymmetrical. Instead of factoring d^{i-1} through $\ker d^i$, we could have factorized d^i through $\operatorname{coker} d^{i-1}$. Then the universal property of coimages induces a morphism $C^{i-1} \to I^i$ making the diagram

$$\begin{tikzcd}
 & & & I^i \arrow[dr] & & \\
\cdots \arrow[r, "d^{i-2}"] & M^{i-1} \arrow[r, "d^{i-1}"] & M^i \arrow[ur, "\operatorname{coim} d^i"] \arrow[dr, "\operatorname{coker} d^{i-1}"'] \arrow[rr] & & M^{i+1} \arrow[r, "d^{i+1}"] & \cdots \\
 & & & C^{i-1} \arrow[uu, dashed] \arrow[ur] & &
\end{tikzcd}$$

commute. Dually to our previous situation, this morphism is always an epimorphism (for $\operatorname{coim} d^i$ is) and is monic if and only if the complex is exact at M^i. Not surprisingly, the source of its kernel is nothing but $H^i(M^\bullet)$.

Proposition 2.1.3 Let $M^\bullet$ be a complex in an abelian category. Then, for every i, there exists a natural morphism $H^i(M^\bullet) \to C^{i-1}$ making the diagram

$$\begin{tikzcd}
 & I^{i-1} \arrow[d] \arrow[dr, "\operatorname{im} d^{i-1}"] & & I^i \arrow[dr] & \\
M^{i-1} \arrow[ur] \arrow[dr] \arrow[rr] & & M^i \arrow[ur, "\operatorname{coim} d^i"] \arrow[dr, "\operatorname{coker} d^{i-1}"'] \arrow[rr] & & M^{i+1} \\
 & K^i \arrow[ur, "\ker d^i"'] \arrow[dr] & & C^{i-1} \arrow[uu] \arrow[ur] & \\
 & & H^i(M^\bullet) \arrow[ur, dashed] & &
\end{tikzcd}$$

commute and satisfying the universal property of the kernel of $C^{i-1} \to I^i$.

Proof. Let $\mu : K^i \to C^{i-1}$ be the composition $\operatorname{coker} d^{i-1} \circ \ker d^i$. By the first isomorphism theorem (theorem 1.2.5), the source of $\operatorname{im} \mu$ and the target of $\operatorname{coim} \mu$ are isomorphic. Thus, it suffices to show that $I^{i-1} \to K^i$ is its kernel and $C^{i-1} \to I^i$ is its cokernel.

The composition $I^{i-1} \to K^i \to C^{i-1}$ is zero, for it coincides with $\operatorname{coker} d^{i-1} \circ \operatorname{im} d^{i-1}$. Moreover, if $\zeta : Z \to K^i$ is another morphism whose composition with μ is zero, then $(\operatorname{coker} d^{i-1}) \circ (\ker d^i) \circ \zeta = 0$ and so the universal property of kernels (using that $\operatorname{im} d^{i-1} = \ker(\operatorname{coker} d^{i-1})$) induces a morphism $Z \to I^{i-1}$ making the diagram commute. This shows that $I^{i-1} \to K^i$ is the kernel of μ. That $C^{i-1} \to I^i$ is its cokernel follows by duality. □

Beyond satisfying our desire for symmetry, the preceding proposition also gives a very useful exact sequence linking cohomologies of different degrees for free.

Corollary 2.1.4 Let $M^\bullet$ be a complex in an abelian category. Then, for every i, the sequence

$$0 \longrightarrow H^i(M^\bullet) \longrightarrow C^{i-1} \longrightarrow K^{i+1} \longrightarrow H^{i+1}(M^\bullet) \longrightarrow 0,$$

where the morphism in the middle is the composition $C^{i-1} \to I^i \to K^{i+1}$, is exact.

Proof. We already know that the sequence is exact at $H^i(M^\bullet)$ and at $H^{i+1}(M^\bullet)$. Exactness at the other objects means that the kernel of $C^{i-1} \to K^{i+1}$ is $H^i(M^\bullet) \to C^{i-1}$ and that its cokernel is $K^{i+1} \to H^{i+1}(M^\bullet)$.

For the first statement, let $Z \to C^{i-1}$ be a morphism whose composition with $C^{i-1} \to K^{i+1}$ is zero. Since the $C^{i-1} \to K^{i+1}$ is the composition of $C^{i-1} \to I^i$ and $I^i \to K^{i+1}$, and the latter is a monomorphism, it follows that $Z \to C^{i-1} \to I^i$ is zero. But then, since $H^i(M^\bullet) \to C^{i-1}$ is the kernel of $C^{i-1} \to I^i$, there's a unique morphism $Z \to H^i(M^\bullet)$ making the diagram commute. The other statement follows in the same way. □

2.2 Exact triangles

One of the main ideas that will motivate our study of homological algebra is the fact that the cohomology functor is not exact, but that somehow we can correct this defect. Let's understand in detail what this means. Consider a short exact sequence of complexes in an abelian category:

$$0 \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow 0.$$

We recall that, since kernels and cokernels on the category of complexes are computed pointwise, this means that all the components

$$0 \longrightarrow L^i \xrightarrow{\varphi^i} M^i \xrightarrow{\psi^i} N^i \longrightarrow 0.$$

are exact. By the functoriality of H^i, we get a complex

$$0 \longrightarrow H^i(L^\bullet) \longrightarrow H^i(M^\bullet) \longrightarrow H^i(N^\bullet) \longrightarrow 0,$$

which is exact at $H^i(M^\bullet)$ but need not be at the extremities. The first statement will emerge as a particular case of our next theorem, but we can see right away that the cohomology functor need not be exact. Indeed, let $L^\bullet = \iota(\mathbb{Z})[-1]$, $M^\bullet$ be the complex whose only non-zero objects are $M^1 = \mathbb{Z}$ and $M^0 = \mathbb{Z}$, and $N^\bullet$ be the complex whose only non-zero objects are $N^1 = \mathbb{Z}/2\mathbb{Z}$ and $N^0 = \mathbb{Z}$. These complexes fit into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\mathrm{id}_{\mathbb{Z}}"] \arrow[d, "\mathrm{id}_{\mathbb{Z}}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact. Then the complex induced by the functoriality of H^0 is

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 2\mathbb{Z} \longrightarrow 0,$$

which is not exact on the right, and the complex induced by H^1 is

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0,$$

which is not exact on the left.

Considering the lack of exactness of H^i, the next best thing we can hope for is to be able to measure how far it is from being exact at each side. Surprisingly, the objects that measure this lack of exactness are the cohomology objects itself shifted in degree. What follows is perhaps the most useful result in homological algebra.

Theorem 2.2.1 — Long exact sequence in cohomology. Consider the following exact sequence of complexes in an abelian category:

$$\begin{tikzcd}
0 \arrow[r] & L^\bullet \arrow[r, "\varphi^\bullet"] & M^\bullet \arrow[r, "\psi^\bullet"] & N^\bullet \arrow[r] & 0.
\end{tikzcd}$$

There exist morphisms $\delta^i : H^i(N^\bullet) \to H^{i+1}(L^\bullet)$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & H^i(L^\bullet) \arrow[r] & H^i(M^\bullet) \arrow[r] & H^i(N^\bullet) \arrow[dll, dashed, "\delta^i"] & \\
& H^{i+1}(L^\bullet) \arrow[r] & H^{i+1}(M^\bullet) \arrow[r] & H^{i+1}(N^\bullet) \arrow[r, dashed] & \cdots
\end{tikzcd}$$

a long exact sequence. The δ^i are said to be *connecting morphisms*.

Proof. First of all, we observe that the snake lemma (theorem 1.6.5) implies that the top row in the diagram

$$\begin{tikzcd}
0 \arrow[r] & K^{i+1}_{L^\bullet} \arrow[r] \arrow[d, "\ker d^{i+1}_{L^\bullet}"] & K^{i+1}_{M^\bullet} \arrow[r] \arrow[d, "\ker d^{i+1}_{M^\bullet}"] & K^{i+1}_{N^\bullet} \arrow[d, "\ker d^{i+1}_{N^\bullet}"] & \\
0 \arrow[r] & L^{i+1} \arrow[r, "\varphi^{i+1}"] \arrow[d, "d^{i+1}_{L^\bullet}"] & M^{i+1} \arrow[r, "\psi^{i+1}"] \arrow[d, "d^{i+1}_{M^\bullet}"] & N^{i+1} \arrow[r] \arrow[d, "d^{i+1}_{N^\bullet}"] & 0 \\
0 \arrow[r] & L^{i+2} \arrow[r, "\varphi^{i+2}"] & M^{i+2} \arrow[r, "\psi^{i+2}"] & N^{i+2} \arrow[r] & 0
\end{tikzcd}$$

is exact. Similarly, it implies that the bottom row in the diagram

$$\begin{tikzcd}
0 \arrow[r] & L^{i-1} \arrow[r, "\varphi^{i-1}"] \arrow[d, "d^{i-1}_{L^\bullet}"] & M^{i-1} \arrow[r, "\psi^{i-1}"] \arrow[d, "d^{i-1}_{M^\bullet}"] & N^{i-1} \arrow[r] \arrow[d, "d^{i-1}_{N^\bullet}"] & 0 \\
0 \arrow[r] & L^{i} \arrow[r, "\varphi^{i}"] \arrow[d, "\text{coker } d^{i-1}_{L^\bullet}"] & M^{i} \arrow[r, "\psi^{i}"] \arrow[d, "\text{coker } d^{i-1}_{M^\bullet}"] & N^{i} \arrow[r] \arrow[d, "\text{coker } d^{i-1}_{N^\bullet}"] & 0 \\
 & C^{i-1}_{L^\bullet} \arrow[r] & C^{i-1}_{M^\bullet} \arrow[r] & C^{i-1}_{N^\bullet} \arrow[r] & 0
\end{tikzcd}$$

is exact.

Now, we fit the morphisms $C^{i-1} \to K^{i+1}$ described in the corollary 2.1.4 into a commutative diagram

$$\begin{tikzcd}
 & C^{i-1}_{L^\bullet} \arrow[r] \arrow[d] & C^{i-1}_{M^\bullet} \arrow[r] \arrow[d] & C^{i-1}_{N^\bullet} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^{i+1}_{L^\bullet} \arrow[r] & K^{i+1}_{M^\bullet} \arrow[r] & K^{i+1}_{N^\bullet},
\end{tikzcd}$$

whose rows are exact. One more application of the snake lemma (theorem 1.6.5) provides the desired connecting morphisms. □

As we argued in the proof of the snake lemma (theorem 1.6.5), even though there may be many morphisms $H^i(N^\bullet) \to H^{i+1}(L^\bullet)$ inducing a long exact sequence, there are distinguished ones which are defined as follows: for a class $[n] \in H^i(N^\bullet)$, let $\hat{n}$ be an element of M^i such that $\psi^i(\hat{n}) = n$. Then $d^i_{M^\bullet}(\hat{n})$ is in the image of φ^{i+1} and we denote its preimage by the same symbol.

$$\begin{tikzcd}
 & \hat{n} \arrow[r, mapsto, "\psi^i"] \arrow[d, mapsto, "d^i_{M^\bullet}"] & n \\
d^i_{M^\bullet}(\hat{n}) \arrow[r, mapsto, "\varphi^{i+1}"] \arrow[d, mapsto] & d^i_{M^\bullet}(\hat{n}) & \\
\delta^i([n]) & &
\end{tikzcd}$$

Finally, δ^i is the map which sends $[n]$ to $[d^i_{M^\bullet}(\hat{n})]$. Whenever we talk about connecting morphisms, it should be understood that these are the morphisms under consideration.

One important property of the connecting morphisms is that they satisfy a certain naturality condition, which we describe below.

Corollary 2.2.2 Consider the following commutative diagram of complexes in an abelian category

$$\begin{tikzcd}
0 \arrow[r] & L^\bullet \arrow[r] \arrow[d] & M^\bullet \arrow[r] \arrow[d] & N^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'^\bullet \arrow[r] & M'^\bullet \arrow[r] & N'^\bullet \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact. Then, for every i, the diagram induced by functoriality and the connecting morphisms

$$\begin{tikzcd}
H^i(N^\bullet) \arrow[r, "\delta^i"] \arrow[d] & H^{i+1}(L^\bullet) \arrow[d] \\
H^i(N'^\bullet) \arrow[r, "\delta'^i"] & H^{i+1}(L'^\bullet)
\end{tikzcd}$$

commutes.

Proof. Let A be the abelian category in question and consider a category $\tilde{A}$, the *arrow category*, whose objects are morphisms in A and whose morphisms between $A \to B$ and $A' \to B'$ are commutative diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & B'.
\end{tikzcd}$$

Since $\tilde{A}$ is nothing but $\mathrm{Fun}(T, A)$, where T is a category with two objects and only one non-trivial morphism between them, the proposition 1.5.2 implies that $\tilde{A}$ is abelian.

A complex in $\tilde{A}$ is nothing but a morphism of complexes in A. Denoting the morphism $L^\bullet \to L'^\bullet$ in $C(A)$ by $\tilde{L}^\bullet$, and similarly for the other morphisms, we obtain a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \tilde{L}^\bullet \arrow[r] & \tilde{M}^\bullet \arrow[r] & \tilde{N}^\bullet \arrow[r] & 0
\end{tikzcd}$$

in $C(\tilde{A})$. Then the previous theorem yields morphisms $\tilde{\delta}^i : H^i(\tilde{N}^\bullet) \to H^{i+1}(\tilde{L}^\bullet)$. Since kernels and cokernels are computed pointwise in a functor category (due to the proof of the aforementioned proposition), a morphism $\tilde{\delta}^i : H^i(\tilde{N}^\bullet) \to H^{i+1}(\tilde{L}^\bullet)$ is nothing but a commuting square as desired. ☐

Due to its somewhat contrived construction, the connecting morphisms doesn't seem to arise in the same fashion as the other morphisms, which are induced from the functoriality of the cohomology functor. This couldn't be further from the truth. We would argue that the long exact sequence in cohomology is simply a shadow of a, perhaps more fundamental, long sequence of complexes. In an ideal world, we would have a morphism of complexes $N^\bullet \to L[1]^\bullet$ and the long exact sequence in cohomology would be nothing but the image of the sequence

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots$$

under the cohomology functor. This doesn't work.[1] The next best thing would be to find a complex $P^\bullet$, along with a quasi-isomorphism $\rho^\bullet : P^\bullet \to N^\bullet$ making the diagram

$$\begin{tikzcd}
H^i(L^\bullet) \arrow[r, "H^i(\varphi^\bullet)"] \arrow[d, equal] & H^i(M^\bullet) \arrow[r, "H^i(\iota^\bullet)"] \arrow[d, equal] & H^i(P^\bullet) \arrow[r, "H^i(\pi^\bullet)"] \arrow[d, "H^i(\rho^\bullet)"] & H^i(L[1]^\bullet) \arrow[d, equal] \\
H^i(L^\bullet) \arrow[r, "H^i(\varphi^\bullet)"] & H^i(M^\bullet) \arrow[r, "H^i(\psi^\bullet)"] & H^i(N^\bullet) \arrow[r, "\delta^i"] & H^{i+1}(L^\bullet)
\end{tikzcd}$$

commute. In this way, the long exact sequence in cohomology arises, up to isomorphism, as the image of

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\iota^\bullet} P^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots$$

under the cohomology functor and connecting morphism δ^i can be described as $H^i(\pi^\bullet) \circ H^i(\rho^\bullet)^{-1}$.

All our hopes and dreams will come true. The reader may recall that there is indeed a natural complex $P^\bullet$ which fits in a short exact sequence

$$0 \longrightarrow M^\bullet \xrightarrow{\iota^\bullet} P^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \longrightarrow 0.$$

It is the direct sum $P^\bullet = M^\bullet \oplus L[1]^\bullet$, with its natural injections and projections. We also have a natural morphism $\rho^\bullet : P^\bullet \to N^\bullet$ defined as the composition of the projection $M^\bullet \oplus L[1]^\bullet \to M^\bullet$ with the given morphism $\psi^\bullet : M^\bullet \to N^\bullet$.

Ay, there's the rub! The natural morphism $\rho^\bullet : P^\bullet \to N^\bullet$ need *not* induce an isomorphism on cohomology. For example, consider the following short exact sequence of complexes of abelian groups

$$0 \longrightarrow \iota(\mathbb{Z}) \xrightarrow{\varphi^\bullet} \iota(\mathbb{Z}) \longrightarrow \iota(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0,$$

where ι is the embedding of Ab into $\mathsf{C}(\mathsf{Ab})$ and $\varphi^\bullet$ is the multiplication by 2 map. In this case, the naive direct sum is simply the complex

$$\cdots \longrightarrow 0 \longrightarrow \iota(\mathbb{Z}) \xrightarrow{0} \underbrace{\iota(\mathbb{Z})}_{\text{degree } 0} \longrightarrow 0 \longrightarrow \cdots$$

---

[1] For example, consider the exact sequence of complexes we used to prove that the cohomology functor is not exact. A morphism of complexes $N^\bullet \to L[1]^\bullet$ inducing the connecting morphism $H^0(N^\bullet) \to H^1(L^\bullet)$ would correspond to a morphism of abelian groups $\mathbb{Z} \to \mathbb{Z}$ whose restriction to $2\mathbb{Z}$ is an isomorphism $2\mathbb{Z} \to \mathbb{Z}$. Such a morphism does not exist.

whose 0-th cohomology is $\mathbb{Z}$, instead of $\mathbb{Z}/2\mathbb{Z}$. The problem, of course, is that our definition of $P^\bullet$ carries no information about morphisms involved in the original exact sequence. If, in the place of the zero-morphism $\iota(\mathbb{Z}) \to \iota(\mathbb{Z})$ above, it was φ^0, no such problem would arise: the 0-th cohomology would be $\mathbb{Z}/2\mathbb{Z}$ and all the other degrees would be zero.

The preceding discussion suggests that it may be useful to consider a complex with the same objects as $M^\bullet \oplus L[1]^\bullet$ but whose i-th differential is given by

$$\begin{pmatrix} d^i_{M^\bullet} & -\varphi[1]^i \\ 0 & d^i_{L[1]^\bullet} \end{pmatrix} = \begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} \\ 0 & -d^{i+1}_{L^\bullet} \end{pmatrix}.$$

As we shall see, it is this object that will solve all our problems.

Definition 2.2.1 — Mapping cone. Let $\varphi^\bullet : L^\bullet \to M^\bullet$ be a morphism of complexes in an additive category. The *mapping cone* of $\varphi^\bullet$ is the complex $\mathrm{MC}(\varphi)^\bullet$ whose objects are $\mathrm{MC}(\varphi)^i := M^i \oplus L^{i+1}$ and whose i-th differential is[a]

$$\begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} \\ 0 & -d^{i+1}_{L^\bullet} \end{pmatrix}.$$

---

[a] There are different sign conventions in the literature.

Since the composition of morphisms represented by matrices is given by the multiplication of the respective matrices, we have that $d^i_{\mathrm{MC}(\varphi)^\bullet} \circ d^{i-1}_{\mathrm{MC}(\varphi)^\bullet}$ is represented by

$$\begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} \\ 0 & -d^{i+1}_{L^\bullet} \end{pmatrix} \begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^{i} \\ 0 & -d^{i}_{L^\bullet} \end{pmatrix} = \begin{pmatrix} d^i_{M^\bullet} \circ d^{i-1}_{M^\bullet} & \varphi^{i+1} \circ d^i_{L^\bullet} - d^i_{M^\bullet} \circ \varphi^i \\ 0 & d^{i+1}_{L^\bullet} \circ d^i_{L^\bullet} \end{pmatrix} = 0,$$

proving that $\mathrm{MC}(\varphi)^\bullet$ is indeed a complex. In an abelian category, the mapping cone inherits a short exact sequence

$$0 \longrightarrow M^\bullet \xrightarrow{\iota^\bullet} \mathrm{MC}(\varphi)^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \longrightarrow 0$$

where the natural injections and projections are still morphisms of complexes, even with the new differential. Moreover, there's an induced long sequence of complexes

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\iota^\bullet} \mathrm{MC}(\varphi)^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots.$$

In order to properly deal with such long sequences of complexes, we introduce some notation. We denote a long sequence of complexes of the form

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\psi^\bullet} N^\bullet \longrightarrow L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots$$

as a *triangle*

$$\begin{tikzcd}
 & L^\bullet \arrow[dr, "\varphi^\bullet"] & \\
N^\bullet \arrow[ur, "+1"] & & M^\bullet, \arrow[ll, "\psi^\bullet"]
\end{tikzcd}$$

where the arrow marked by $+1$ indicates that the morphism shifts the degree by one, representing the imposing diagram

$$\begin{tikzcd}
 & L^{i+1} \arrow[dr] & \\
N^{i+1} & & M^{i+1} \arrow[ll] \\
 & L^{i} \arrow[uu, "d^i_{N^\bullet}"] \arrow[dr] & \\
N^{i} \arrow[uu, "d^i_{N^\bullet}"] \arrow[uuur] & & M^{i} \arrow[ll] \arrow[uu, "d^i_{M^\bullet}"'] \\
 & L^{i-1} \arrow[uu, "d^{i-1}_{N^\bullet}"] \arrow[dr] & \\
N^{i-1} \arrow[uu, "d^{i-1}_{N^\bullet}"] \arrow[uuur] & & M^{i-1}. \arrow[ll] \arrow[uu, "d^{i-1}_{M^\bullet}"']
\end{tikzcd}$$

A morphism of triangles consists of morphisms $\lambda^\bullet$, $\mu^\bullet$, and $\nu^\bullet$, making the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r, "\varphi^\bullet"] \arrow[d, "\lambda^\bullet"] & M^\bullet \arrow[r, "\psi^\bullet"] \arrow[d, "\mu^\bullet"] & N^\bullet \arrow[r] \arrow[d, "\nu^\bullet"] & L[1]^\bullet \arrow[d, "\lambda[1]^\bullet"] \\
L'^\bullet \arrow[r, "\varphi'^\bullet"] & M'^\bullet \arrow[r, "\psi'^\bullet"] & N'^\bullet \arrow[r] & L'[1]^\bullet
\end{tikzcd}$$

commute. Moreover, a triangle is said to be exact if it arises from a long exact sequence.[2]

In this notation, the plan we outlined before can be encapsulated as the fact that, given a short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & L^\bullet \arrow[r, "\varphi^\bullet"] & M^\bullet \arrow[r, "\psi^\bullet"] & N^\bullet \arrow[r] & 0,
\end{tikzcd}$$

the cohomology functor $H^\bullet$ takes the triangle induced by $\mathrm{MC}(\varphi)^\bullet$ and outputs an exact triangle

$$\begin{tikzcd}
 & H^\bullet(L^\bullet) \arrow[dr, "H^\bullet(\varphi^\bullet)"] & \\
H^\bullet(\mathrm{MC}(\varphi)^\bullet) \arrow[ur, "+1"] & & H^\bullet(M^\bullet), \arrow[ll]
\end{tikzcd}$$

---

[2] Some references define an *exact triangle* to be what we'll soon call a *distinguished triangle*.

which is isomorphic to the triangle arising from the long exact sequence in cohomology

$$\begin{tikzcd}
& H^\bullet(L^\bullet) \arrow[dr, "H^\bullet(\varphi^\bullet)"] & \\
H^\bullet(N^\bullet) \arrow[ur, "+1"] & & H^\bullet(M^\bullet). \arrow[ll, "H^\bullet(\psi^\bullet)"']
\end{tikzcd}$$

We now prove this fact.

Proposition 2.2.3 Consider the following exact sequence of complexes in an abelian category:

$$\begin{tikzcd}
0 \arrow[r] & L^\bullet \arrow[r, "\varphi^\bullet"] & M^\bullet \arrow[r, "\psi^\bullet"] & N^\bullet \arrow[r] & 0.
\end{tikzcd}$$

There exists a quasi-isomorphism $\rho^\bullet : \mathrm{MC}(\varphi)^\bullet \to N^\bullet$ making the diagram

$$\begin{tikzcd}
H^{i-1}(L[1]^\bullet) \arrow[r] \arrow[d, equal] & H^i(M^\bullet) \arrow[r, "H^i(\iota^\bullet)"] \arrow[d, equal] & H^i(\mathrm{MC}(\varphi)^\bullet) \arrow[r, "H^i(\pi^\bullet)"] \arrow[d, "H^i(\rho^\bullet)"] & H^i(L[1]^\bullet) \arrow[d, equal] \\
H^i(L^\bullet) \arrow[r, "H^i(\varphi^\bullet)"] & H^i(M^\bullet) \arrow[r, "H^i(\psi^\bullet)"] & H^i(N^\bullet) \arrow[r, "\delta^i"] & H^{i+1}(L^\bullet)
\end{tikzcd}$$

commute.

Proof. As before, $\rho^\bullet$ is simply the composition of the projection $\mathrm{MC}(\varphi)^\bullet \to M^\bullet$ with the given morphism $\psi^\bullet : M^\bullet \to N^\bullet$. This morphism makes the middle square commute due to the fact that the composition $M^\bullet \to \mathrm{MC}(\varphi)^\bullet \to M^\bullet$ is the identity (theorem 1.1.9) and so the diagram below

$$\begin{tikzcd}
M^\bullet \arrow[r, "\iota^\bullet"] \arrow[dd, equal] & \mathrm{MC}(\varphi)^\bullet \arrow[d] \arrow[dd, bend left, "\rho^\bullet"] \\
& M^\bullet \arrow[d, "\psi^\bullet"'] \\
M^\bullet \arrow[r, "\psi^\bullet"] & N^\bullet
\end{tikzcd}$$

commutes. The commutativity of the square on the right means that the composition $\delta^i \circ H^i(\rho^\bullet)$ sends $[(m, l)] \in H^i(\mathrm{MC}(\varphi)^\bullet)$ to $[l] \in H^{i+1}(L^\bullet)$. Now, $H^i(\rho^\bullet)$ sends $[(m, l)]$ to $[\psi^i(m)]$ and δ^i sends this element to $[l']$, where l' is some element satisfying $\varphi^{i+1}(l') = d^i_{M^\bullet}(m)$. But l itself is one such element, due to the fact that $(m, l) \in \ker d^i_{\mathrm{MC}(\varphi)^\bullet}$.

All that remains is to prove that $H^i(\rho^\bullet)$ is an isomorphism for all i. But, upon extending our diagram one square to the right, this follows directly from the five lemma (proposition 1.6.2). □

Beyond being conceptually useful, the last proposition also provides us with a criterion of a morphism of complexes to be a quasi-isomorphism.

Corollary 2.2.4 Let $\varphi^\bullet : L^\bullet \to M^\bullet$ be a morphism of complexes. Then $\varphi^\bullet$ is a quasi-isomorphism if and only if its mapping cone $\mathrm{MC}(\varphi)^\bullet$ is an exact complex.

Proof. If $\mathrm{MC}(\varphi)^\bullet$ is an exact complex, then its cohomology is zero and so the long exact sequence in cohomology

$$\underbrace{H^{i-1}(\mathrm{MC}(\varphi)^\bullet)}_{=0} \longrightarrow H^i(L^\bullet) \xrightarrow{H^i(\varphi^\bullet)} H^i(M^\bullet) \longrightarrow \underbrace{H^i(\mathrm{MC}(\varphi)^\bullet)}_{=0}$$

implies that $\varphi^\bullet$ is a quasi-isomorphism. Conversely, if $\varphi^\bullet$ is a quasi-isomorphism, the long exact sequence in cohomology

$$H^i(L^\bullet) \xrightarrow{H^i(\varphi^\bullet)} H^i(M^\bullet) \xrightarrow{\alpha} H^i(\mathrm{MC}(\varphi)^\bullet) \xrightarrow{\beta} H^{i+1}(L^\bullet) \xrightarrow{H^{i+1}(\varphi^\bullet)} H^{i+1}(M^\bullet)$$

implies that $\ker\alpha = \mathrm{id}_{H^i(M^\bullet)}$, that $\mathrm{im}\,\alpha = \ker\beta$ and that $\mathrm{im}\,\beta$ is the zero-morphism. The first and the last pieces of information mean that both α and β are zero-morphisms, and $\mathrm{im}\,\alpha = \ker\beta$ implies that β is a monomorphism. But then $\ker\beta$ is both the identity on $H^i(\mathrm{MC}(\varphi)^\bullet)$ and the zero-morphism $0 \to H^i(\mathrm{MC}(\varphi)^\bullet)$. It follows that $H^i(\mathrm{MC}(\varphi)^\bullet) = 0$. ☐

This corollary allows us to prove that quasi-isomorphisms are preserved by exact functors.

Corollary 2.2.5 Let $F : A \to B$ be an exact functor between abelian categories and $\varphi^\bullet : L^\bullet \to M^\bullet$ be a morphism of complexes in A. If $\varphi^\bullet$ is a quasi-isomorphism, then so is $F(\varphi^\bullet)$.

Proof. Due to the last corollary, it suffices to prove that $\mathrm{MC}(F(\varphi^\bullet)) = F(\mathrm{MC}(\varphi^\bullet))$ is an exact complex. But the proposition 2.1.2 implies that

$$H^\bullet(F(\mathrm{MC}(\varphi^\bullet))) = F(\underbrace{H^\bullet(\mathrm{MC}(\varphi^\bullet))}_{=0}) = 0,$$

establishing the result. ☐

One aspect of mapping cone of a morphism $\varphi^\bullet : L^\bullet \to M^\bullet$ that we have not yet addressed is the fact that, even though $\mathrm{MC}(\varphi)^\bullet$ is always a complex, the long sequence induced

$$\cdots \longrightarrow L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet \xrightarrow{\iota^\bullet} \mathrm{MC}(\varphi)^\bullet \xrightarrow{\pi^\bullet} L[1]^\bullet \xrightarrow{\varphi[1]^\bullet} M[1]^\bullet \longrightarrow \cdots$$

need not be. This means that our triangles aren't elements of $C(C(A))$. Indeed, the composition

$$L^\bullet \to M^\bullet \to \mathrm{MC}(\varphi)^\bullet$$

sends $l \in L^i$ to $(\varphi^i(l), 0) \in M^i \oplus L^{i+1}$, which isn't always zero unless φ^i is the zero morphism. Moreover, the composition

$$\mathrm{MC}(\varphi)^\bullet \to L[1]^\bullet \to M[1]^\bullet$$

sends $(m, l) \in M^i \oplus L^{i+1}$ to $\varphi^{i+1}(l) \in M^{i+1}$, which also isn't always zero unless φ^{i+1} is the zero morphism.

Notwithstanding the fact that these compositions are usually not zero, they do indeed map to the zero-morphism in cohomology. And they do so for a good reason, which will be the main focus of the next section.

2.3 The homotopic category

The main line of attack in homological algebra to understanding some mathematical object consists of associating some interesting complex to this object and then taking its cohomology. For example, given a smooth manifold M, we associate to it a complex

$$0 \longrightarrow \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots,$$

where Ω^i_M is the $\mathbb{R}$-vector space of differential i-forms on M and d is the exterior derivative. The i-th cohomology of this complex $H^i_{dR}(M)$ is said to be the *de Rham cohomology* of M and is an important invariant of a manifold. Of a more algebraic nature are the modules $\mathrm{Tor}^A_i(M, N)$ which are computed in the following way: we find an exact sequence of A-modules

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is a *projective* module, we tensor by N and take the $-i$-th cohomology of the complex

$$\cdots \longrightarrow P_3 \otimes_A N \longrightarrow P_2 \otimes_A N \longrightarrow P_1 \otimes_A N \longrightarrow P_0 \otimes_A N.$$

(All the omitted objects are supposed to be zero.) Surprisingly, the final result is independent of the choice of the projective modules P_i. We could even take the P_i to be flat modules and the result wouldn't change.

But we can do better! Instead of taking the cohomology of the associated complexes, we can consider them "up to quasi-isomorphism". In this way we retain all the cohomological information while being able to use the tools available for dealing with complexes. Somewhat more formally, we would like to find a category $\mathsf{D}(A)$, with the same objects as $\mathsf{C}(A)$ but where all the quasi-isomorphisms become genuine isomorphisms.

This category, along with its bounded variants $\mathsf{D}^*(\mathsf{A})$ for $* = +, -, b$, indeed exists[3] and it's called the *derived category* of A. This category satisfies a universal property alike that of the localization of modules: it is endowed with an additive functor $\mathsf{C}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$ such that quasi-isomorphisms in $\mathsf{C}(\mathsf{A})$ are mapped to isomorphisms in $\mathsf{D}(\mathsf{A})$ and which is initial with respect to this property.

It is the derived category that is the natural place to study homological algebra. Nevertheless, there is an intermediate category, the *homotopic category*, that will not only simplify the description of the morphisms in the derived category but also furnish a substitute thereof in important cases. We begin its study now.

Definition 2.3.1 — Homotopy. Let $\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$ be two morphisms of complexes in an additive category. A *homotopy* between $\varphi^\bullet$ and $\psi^\bullet$ is a collection of morphisms $h^i : L^i \to M^{i-1}$ such that

$$\psi^i - \varphi^i = d_{M^\bullet}^{i-1} \circ h^i + h^{i+1} \circ d_{L^\bullet}^i.$$

for all i. If there exists a homotopy between $\varphi^\bullet$ and $\psi^\bullet$, we say that they are *homotopic*, and we denote it by $\varphi^\bullet \sim \psi^\bullet$.

We observe that this is indeed an equivalence relation: reflexivity and symmetry are immediate, and it suffices to sum the homotopies to prove that it is transitive. We also emphasize that the h^i need not form a morphism of complexes $L^\bullet \to M[-1]^\bullet$. In particular, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & L^{i-1} \arrow[r, "d_{L^\bullet}^{i-1}"] \arrow[d, shift right, "\varphi^{i-1}"'] \arrow[d, shift left, "\psi^{i-1}"] \arrow[dl] & L^i \arrow[r, "d_{L^\bullet}^{i}"] \arrow[d, shift right, "\varphi^{i}"'] \arrow[d, shift left, "\psi^{i}"] \arrow[dl, "h^i"'] & L^{i+1} \arrow[r] \arrow[d, shift right, "\varphi^{i+1}"'] \arrow[d, shift left, "\psi^{i+1}"] \arrow[dl, "h^{i+1}"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & M^{i-1} \arrow[r, "d_{M^\bullet}^{i-1}"'] & M^i \arrow[r, "d_{M^\bullet}^{i}"'] & M^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

need not commute. The next proposition explains how homotopy interacts with the additive structure of $\mathsf{C}(\mathsf{A})$. The reader may remember its first part as saying that "morphisms homotopic to zero form an ideal".

Proposition 2.3.1 Let $\varphi_1^\bullet, \varphi_2^\bullet : L^\bullet \to M^\bullet$ and $\psi_1^\bullet, \psi_2^\bullet : M^\bullet \to N^\bullet$ be morphisms of complexes in an additive category. The following holds.

(a) If $\varphi_1^\bullet \sim 0$ and $\varphi_2^\bullet \sim 0$, then $\varphi_1^\bullet + \varphi_2^\bullet \sim 0$, $\varphi_1^\bullet \circ \alpha^\bullet \sim 0$ and $\beta^\bullet \circ \varphi_1^\bullet \sim 0$ whenever those compositions exist;

(b) if $\varphi_1^\bullet \sim \varphi_2^\bullet$ and $\psi_1^\bullet \sim \psi_2^\bullet$, then $\psi_1^\bullet \circ \varphi_1^\bullet \sim \psi_2^\bullet \circ \varphi_2^\bullet$.

Proof. If $\varphi_1^\bullet \sim 0$ and $\varphi_2^\bullet \sim 0$, then there exists collections of morphisms $h^i, k^i : L^i \to$

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[3] Up to some set-theoretic subtleties, which will be discussed in the next chapter.

M^{i-1} such that

$$\varphi_1^i = d_{M^\bullet}^{i-1} \circ h^i + h^{i+1} \circ d_{L^\bullet}^i \quad \text{and} \quad \varphi_2^i = d_{M^\bullet}^{i-1} \circ k^i + k^{i+1} \circ d_{L^\bullet}^i.$$

Summing these equations, we see that the morphisms $h^i + k^i$ form a homotopy between $\varphi_1^\bullet + \varphi_2^\bullet$ and zero. By composing on the left with $\alpha^\bullet : P^\bullet \to L^\bullet$ we get that

$$\begin{aligned} \varphi_1^i \circ \alpha^i &= d_{M^\bullet}^{i-1} \circ h^i \circ \alpha^i + h^{i+1} \circ d_{L^\bullet}^i \circ \alpha^i \\ &= d_{M^\bullet}^{i-1} \circ (h^i \circ \alpha^i) + (h^{i+1} \circ \alpha^{i+1}) \circ d_{P^\bullet}^i, \end{aligned}$$

proving that $h^i \circ \alpha^i$ is a homotopy between $\varphi_1^\bullet \circ \alpha^\bullet$ and 0. The same argument proves that $\beta^\bullet \circ \varphi_1^\bullet \sim 0$. This establishes (a).

Now, (b) follows from (a) by noticing that

$$\begin{aligned} 0 &\sim \psi_1^\bullet \circ (\varphi_1^\bullet - \varphi_2^\bullet) = \psi_1^\bullet \circ \varphi_1^\bullet - \psi_1^\bullet \circ \varphi_2^\bullet \\ 0 &\sim (\psi_1^\bullet - \psi_2^\bullet) \circ \varphi_1^\bullet = \psi_1^\bullet \circ \varphi_1^\bullet - \psi_2^\bullet \circ \varphi_1^\bullet \end{aligned}$$

and subtracting the two equations. ☐

There's an important definition which encodes the notion of "isomorphism up to homotopy".

Definition 2.3.2 — Homotopy equivalence. A morphism of complexes $\varphi^\bullet : L^\bullet \to M^\bullet$ is said to be a *homotopy equivalence* if there exists a morphism $\psi^\bullet : M^\bullet \to L^\bullet$ such that $\varphi^\bullet \circ \psi^\bullet \sim \mathrm{id}_{M^\bullet}$ and $\psi^\bullet \circ \varphi^\bullet \sim \mathrm{id}_{L^\bullet}$. If there exists a homotopy equivalence between two complexes, they are said to be *homotopy equivalent*.

Once again, this defines an equivalence relation: reflexivity and symmetry are clear and transitivity follows from the preceding proposition. We observe that, from this point of view, the notion of homotopy equivalence is better behaved then that of quasi-isomorphism as the latter doesn't define an equivalence relation between complexes. Indeed, in $\mathsf{C}(\mathsf{Ab})$ the morphism of complexes

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

is a quasi-isomorphism which does not possess an inverse (as there are no non-trivial morphisms $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$), proving that quasi-isomorphism is not a symmetric relation.

Even though the above is only one of the multiple reasons why homotopy equivalence is a more tractable notion than that of quasi-isomorphisms, it would all be for nothing if homotopy weren't a stepping stone to the derived category. The next proposition begins to describe how our plan works.

Proposition 2.3.2 Let $\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$ be homotopic morphisms of complexes in an abelian category. Then $\varphi^\bullet$ and $\psi^\bullet$ induce the same morphism on cohomology.

Proof. We prove that $\psi^\bullet - \varphi^\bullet$ induces the zero-morphism in cohomology, i.e., that it sends elements of $\ker d^i_{L^\bullet}$ to elements of $\operatorname{im} d^{i-1}_{M^\bullet}$. But this is clear since

$$\psi^i(l) - \varphi^i(l) = d^{i-1}_{M^\bullet}(h^i(l)) + h^{i+1}(d^i_{L^\bullet}(l))$$

and the last term vanishes whenever $l \in \ker d^i_{L^\bullet}$. ☐

This is why the long sequence induced by the mapping cone of a morphism "has a good reason" to become a complex in cohomology: the composition of two consecutive morphisms is not necessarily zero, but they are *homotopic* to zero. This proposition, in the form of the corollary below, also describes why the line of attack described in the beginning of this section works: often we'll associate non-isomorphic complexes to a mathematical object, but they'll turn out to be homotopy equivalent.

Corollary 2.3.3 Let $L^\bullet$ and $M^\bullet$ be homotopy equivalent complexes in an abelian category. Then $H^\bullet(L^\bullet) \cong H^\bullet(M^\bullet)$.

Proof. Due to the preceding proposition, the morphisms which define a homotopy equivalence between $L^\bullet$ and $M^\bullet$ induce inverse morphisms in cohomology. ☐

There is another aspect where homotopy equivalences are simpler than quasi-isomorphisms: while the latter is only[4] preserved by an exact functor (corollary 2.2.5), the former is preserved by arbitrary additive functors.

Proposition 2.3.4 Let $F : A \to B$ be an additive functor between additive categories, and let $\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$ be homotopic morphisms in $C(A)$. Then $F(\varphi^\bullet)$ and $F(\psi^\bullet)$ are homotopic in $C(B)$. Moreover, if $L^\bullet$ and $M^\bullet$ are homotopy equivalent in $C(A)$, then $F(L^\bullet)$ and $F(M^\bullet)$ are homotopy equivalent in $C(B)$.

Proof. The second assertion follows immediately from the first. As for the first, observe that if h is a homotopy between $\varphi^\bullet$ and $\psi^\bullet$, then

$$\begin{aligned} F(\psi^i) - F(\varphi^i) &= F(d^{i-1}_{M^\bullet}) \circ F(h^i) + F(h^{i+1}) \circ F(d^i_{L^\bullet}) \\ &= d^{i-1}_{F(M^\bullet)} \circ F(h^i) + F(h^{i+1}) \circ d^i_{F(L^\bullet)}. \end{aligned}$$

This proves that the morphisms $F(h^i)$ define a homotopy between $F(\varphi^\bullet)$ and $F(\psi^\bullet)$. ☐

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[4] Indeed, if F is not exact, there exists a three-term exact sequence whose image is not exact. But a three-term complex is exact if and only if it is quasi-isomorphic to zero.

This validates our strategy: beginning with a mathematical object to which we associate some interesting complex, we apply some additive functor and then see the result in the derived category. The proposition 2.3.4 and the corollary 2.3.3 shows that any other homotopy equivalent complex would yield the same result at the end.

By identifying homotopic morphisms, we obtain the homotopic category.

Definition 2.3.3 — Homotopic category. Let A be an additive category. The *homotopic category* $\mathsf{K}(\mathsf{A})$ is the category whose objects are complexes in A and whose morphisms are homotopy classes of morphisms of complexes. We define likewise bounded variants $\mathsf{K}^*(\mathsf{A})$, for $* = +, -, \mathfrak{b}$, thereof.

The part (b) of proposition 2.3.1 implies that indeed $\mathsf{K}^*(\mathsf{A})$ satisfy the axioms of a category, and the part (a) shows that they are moreover preadditive. Since they have a zero-object and binary products, they are even additive. They aren't, through, almost never abelian even if A is. Indeed, we'll soon see that if $\mathsf{K}^*(\mathsf{A})$ is abelian then every short exact sequence in A splits.

We observe that if $F : \mathsf{A} \to \mathsf{B}$ is a functor between additive categories, then we have a natural functor $F : \mathsf{K}(\mathsf{A}) \to \mathsf{K}(\mathsf{B})$ by the proposition 2.3.4 and the universal property of quotients.

The reader may recall that our long-term goal is to understand the derived category, which is constructed from $\mathsf{C}(\mathsf{A})$ by inverting all the quasi-isomorphisms. In defining the homotopic category, we have determined a functor

$$\mathsf{C}(\mathsf{A}) \to \mathsf{K}(\mathsf{A})$$

which sends every object to itself and every morphism to its homotopy class. This functor sends every homotopy equivalence to an isomorphism and, as the proposition below shows, is a stepping stone to the derived category.

Proposition 2.3.5 Let $F : \mathsf{C}^*(\mathsf{A}) \to \mathsf{D}$ be an additive functor such that $F(\varphi^\bullet)$ is an isomorphism whenever $\varphi^\bullet$ is a quasi-isomorphism. Then there exists a unique additive functor $\mathsf{K}^*(\mathsf{A}) \to \mathsf{D}$ making the diagram

$$\begin{tikzcd}
\mathsf{C}^*(\mathsf{A}) \arrow[r, "F"] \arrow[d] & \mathsf{D} \\
\mathsf{K}^*(\mathsf{A}) \arrow[ur, dashed] &
\end{tikzcd}$$

commute.

Proof. We need to show that if $\varphi^\bullet, \psi^\bullet : L^\bullet \to M^\bullet$ are homotopic morphisms in $\mathsf{C}^*(\mathsf{A})$, then $F(\varphi^\bullet) = F(\psi^\bullet)$. Since F is additive, we may assume $\psi^\bullet = 0$. Moreover, as $-\mathrm{id}_{L^\bullet}$ is a quasi-isomorphism, the corollary 2.2.4 implies that it suffices to prove that $\varphi^\bullet$ factors through $\mathrm{MC}(-\mathrm{id}_{L^\bullet})^\bullet$.

We already possess the natural injection $L^\bullet \to \mathrm{MC}(-\mathrm{id}_{L^\bullet})^\bullet$ so we only have to define a morphism of complexes $\pi^\bullet : \mathrm{MC}(-\mathrm{id}_{L^\bullet})^\bullet \to M^\bullet$ making the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r, "\varphi^\bullet"] \arrow[d] & M^\bullet \\
\mathrm{MC}(-\mathrm{id}_{L^\bullet})^\bullet \arrow[ur, "\pi^\bullet"'] &
\end{tikzcd}$$

commute. With that in mind, consider a homotopy $h^i : L^i \to M^{i-1}$ between $\varphi^\bullet$ and the zero-morphism. We then define our desired morphism $\pi^i : L^i \oplus L^{i+1} \to M^i$ as (φ^i, h^{i+1}). It is clear that this makes the diagram above commute. It being a morphism of complexes means that

$$\begin{aligned}
\begin{pmatrix} d^{i-1}_{M^\bullet} \circ \varphi^{i-1} & d^{i-1}_{M^\bullet} \circ h^i \end{pmatrix} &= \begin{pmatrix} \varphi^i & h^{i+1} \end{pmatrix} \begin{pmatrix} d^{i-1}_{L^\bullet} & \mathrm{id}_{L^i} \\ 0 & -d^i_{L^\bullet} \end{pmatrix} \\
&= \begin{pmatrix} \varphi^i \circ d^{i-1}_{L^\bullet} & \varphi^i - h^{i+1} \circ d^i_{L^\bullet} \end{pmatrix}
\end{aligned}$$

is verified for all i. The first equation holds due to the fact that $\varphi^\bullet : L^\bullet \to M^\bullet$ is a morphism of complexes, and the second holds as the h^i define a homotopy between $\varphi^\bullet$ and zero. This completes the proof. □

Since the cohomology functors $H^i : \mathsf{C}^*(\mathsf{A}) \to \mathsf{A}$ send quasi-isomorphisms to bona fide isomorphisms, they descend to well-defined functors $\mathsf{K}^*(\mathsf{A}) \to \mathsf{A}$ which will still be denoted by H^i. In particular, it makes sense to ask whether a morphism in $\mathsf{K}^*(\mathsf{A})$ is a quasi-isomorphism or not, and we can construct the derived category by inverting the quasi-isomorphisms *in the homotopic category*. This will turn out to be simpler than going straight from $\mathsf{C}^*(\mathsf{A})$.

2.4 The triangulated structure

As hinted before, even if A is an abelian category, the homotopic category $\mathsf{K}^*(\mathsf{A})$ need not be. So, in order to be able to do homological algebra, we need some sort of substitute in $\mathsf{K}^*(\mathsf{A})$ for short exact sequences. It turns out that triangles behave even better in $\mathsf{K}^*(\mathsf{A})$ than they do in $\mathsf{C}^*(\mathsf{A})$.

The shift functor $[n] : \mathsf{C}^*(\mathsf{A}) \to \mathsf{C}^*(\mathsf{A})$ preserves homotopies and so descends to a functor $\mathsf{K}^*(\mathsf{A}) \to \mathsf{K}^*(\mathsf{A})$ denoted by the same symbol. As before, a *triangle* in $\mathsf{K}^*(\mathsf{A})$

$$\begin{tikzcd}
& L^\bullet \arrow[dr, "\overline{\varphi}"] & \\
N^\bullet \arrow[ur, "+1"] & & M^\bullet, \arrow[ll, "\overline{\psi}"]
\end{tikzcd}$$

or even

$$L^\bullet \xrightarrow{\overline{\varphi}} M^\bullet \xrightarrow{\overline{\psi}} N^\bullet \longrightarrow L[1]^\bullet,$$

is a shorthand for a long sequence of the form

$$\cdots \longrightarrow L^\bullet \xrightarrow{\overline{\varphi}} M^\bullet \xrightarrow{\overline{\psi}} N^\bullet \longrightarrow L[1]^\bullet \xrightarrow{\overline{\varphi}[1]} M[1]^\bullet \longrightarrow \cdots,$$

where the morphisms involved are now those of $\mathsf{K}^*(\mathsf{A})$. Similarly, a morphism of triangles consists of morphisms $\overline{\lambda}$, $\overline{\mu}$, and $\overline{\nu}$ in $\mathsf{K}^*(\mathsf{A})$, making the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\lambda}"] & M^\bullet \arrow[r, "\overline{\psi}"] \arrow[d, "\overline{\mu}"] & N^\bullet \arrow[r] \arrow[d, "\overline{\nu}"] & L[1]^\bullet \arrow[d, "\overline{\lambda}[1]"] \\
L'^\bullet \arrow[r, "\overline{\varphi}'"] & M'^\bullet \arrow[r, "\overline{\psi}'"] & N'^\bullet \arrow[r] & L'[1]^\bullet
\end{tikzcd}$$

commute. As in $\mathsf{C}^*(\mathsf{A})$, the triangles arising from mapping cones have a prominent role.

Definition 2.4.1 — Distinguished triangles. A triangle in $\mathsf{K}^*(\mathsf{A})$ is said to be *distinguished* if it is isomorphic to some triangle of the form

$$\begin{tikzcd}
& L^\bullet \arrow[dr, "\overline{\varphi}"] & \\
\mathrm{MC}(\varphi)^\bullet \arrow[ur, "+1"] & & M^\bullet \arrow[ll]
\end{tikzcd}$$

for a morphism $\varphi^\bullet : L^\bullet \to M^\bullet$ in $\mathsf{C}^*(\mathsf{A})$.

As a first sign that triangles work better in $\mathsf{K}^*(\mathsf{A})$ than they do in $\mathsf{C}^*(\mathsf{A})$, we observe that the identity morphism always defines a distinguished triangle in $\mathsf{K}^*(\mathsf{A})$. This means that even though the mapping cone of the identity morphism is not zero, it is homotopy equivalent to zero.

Lemma 2.4.1 Let $M^\bullet$ be a complex in an additive category A. Then the triangle

$$\begin{tikzcd}
& M^\bullet \arrow[dr, "\overline{\mathrm{id}_{M^\bullet}}"] & \\
0 \arrow[ur, "+1"] & & M^\bullet \arrow[ll]
\end{tikzcd}$$

is distinguished.

Proof. Consider the collection of morphisms $h^i : M^i \oplus M^{i+1} \to M^{i-1} \oplus M^i$ given by the matrices

$$\begin{pmatrix} 0 & 0 \\ -\mathrm{id}_{M^i} & 0 \end{pmatrix}.$$

The composition $d^{i-1}_{\mathrm{MC}(\mathrm{id}_{M^\bullet})^\bullet} \circ h^i + h^{i+1} \circ d^i_{\mathrm{MC}(\mathrm{id}_{M^\bullet})^\bullet}$ is represented by the matrix

$$\begin{pmatrix} d^{i-1}_{M^\bullet} & -\mathrm{id}_{M^i} \\ 0 & -d^i_{M^\bullet} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\mathrm{id}_{M^i} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} d^i_{M^\bullet} & -\mathrm{id}_{M^{i+1}} \\ 0 & -d^{i+1}_{M^\bullet} \end{pmatrix},$$

which is nothing but the identity of $\mathrm{MC}(\mathrm{id}_{M^\bullet})^i$. It follows that the identity morphism on $\mathrm{MC}(\mathrm{id}_{M^\bullet})^\bullet$ is homotopic to zero, and so the natural morphism $0 \to \mathrm{MC}(\mathrm{id}_{M^\bullet})^\bullet$ is an isomorphism in $\mathsf{K}^*(\mathsf{A})$. □

Another useful property of distinguished triangles in $\mathsf{K}^*(\mathsf{A})$ is that they remain distinguished upon rotation. We observe that, while the proof is basically only the definition of a homotopy equivalence, there are a lot of things that need to be verified, and we won't shy away.

Lemma 2.4.2 Let A be an additive category. Consider the following triangles in $\mathsf{K}^*(\mathsf{A})$:

$$\begin{tikzcd} & L^\bullet \arrow[dr, "\overline{\varphi}"] & \\ N^\bullet \arrow[ur, "\overline{\rho}", "+1"'] & & M^\bullet \arrow[ll, "\overline{\psi}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & M^\bullet \arrow[dr, "\overline{\psi}"] & \\ L[1]^\bullet \arrow[ur, "-\overline{\varphi}[1]^\bullet", "+1"'] & & N^\bullet. \arrow[ll, "\overline{\rho}"] \end{tikzcd}$$

Then one of the triangles is distinguished if and only if the other is.

Proof. We first suppose that the triangle on the left is of the form

$$\begin{tikzcd} & L^\bullet \arrow[dr, "\overline{\varphi}"] & \\ \mathrm{MC}(\varphi)^\bullet \arrow[ur, "+1"] & & M^\bullet. \arrow[ll, "\overline{\iota}"] \end{tikzcd}$$

Since the mapping cone of $\iota^\bullet : M^\bullet \to \mathrm{MC}(\varphi)^\bullet$ is naturally endowed with morphisms $\mathrm{MC}(\varphi)^\bullet \to \mathrm{MC}(\iota)^\bullet$ and $\mathrm{MC}(\iota)^\bullet \to M[1]^\bullet$, it suffices to prove the existence of a homotopy equivalence $L[1]^\bullet \to \mathrm{MC}(\iota)^\bullet$ making the diagram

$$\begin{tikzcd} M^\bullet \arrow[r, "\overline{\iota}"] \arrow[d, equal] & \mathrm{MC}(\varphi)^\bullet \arrow[r] \arrow[d, equal] & L[1]^\bullet \arrow[r, "-\overline{\varphi}[1]^\bullet"] \arrow[d, dashed] & M[1]^\bullet \arrow[d, equal] \\ M^\bullet \arrow[r, "\overline{\iota}"] & \mathrm{MC}(\varphi)^\bullet \arrow[r] & \mathrm{MC}(\iota)^\bullet \arrow[r] & M[1]^\bullet \end{tikzcd}$$

commute for the triangle on the right to be distinguished. We define a morphism $L^{i+1} \to M^i \oplus L^{i+1} \oplus M^{i+1}$ by

$$\begin{pmatrix} 0 \\ \mathrm{id}_{L^{i+1}} \\ -\varphi^{i+1} \end{pmatrix}.$$

Also, a morphism $M^i \oplus L^{i+1} \oplus M^{i+1} \to L^{i+1}$ is given by projecting onto the middle coordinate. The composition $L^{i+1} \to M^i \oplus L^{i+1} \oplus M^{i+1} \to L^{i+1}$ is

$$\begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id}_{L^{i+1}} \\ -\varphi^{i+1} \end{pmatrix} = \mathrm{id}_{L^{i+1}}$$

and the composition $M^i \oplus L^{i+1} \oplus M^{i+1} \to L^{i+1} \to M^i \oplus L^{i+1} \oplus M^{i+1}$,

$$\begin{pmatrix} 0 \\ \mathrm{id}_{L^{i+1}} \\ -\varphi^{i+1} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+1}} & 0 \\ 0 & -\varphi^{i+1} & 0 \end{pmatrix},$$

is homotopic to the identity via the homotopy $h^i : M^i \oplus L^{i+1} \oplus M^{i+1} \to M^{i-1} \oplus L^i \oplus M^i$ given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{M^i} & 0 & 0 \end{pmatrix}.$$

Indeed, the morphism $d^{i-1}_{\mathrm{MC}(\iota)^\bullet} \circ h^i + h^{i+1} \circ d^i_{\mathrm{MC}(\iota)^\bullet}$ is represented by

$$\begin{aligned} &\begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i & -\mathrm{id}_{M^i} \\ 0 & -d^i_{L^\bullet} & 0 \\ 0 & 0 & -d^i_{M^\bullet} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{M^i} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{id}_{M^{i+1}} & 0 & 0 \end{pmatrix} \begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} & -\mathrm{id}_{M^{i+1}} \\ 0 & -d^{i+1}_{L^\bullet} & 0 \\ 0 & 0 & -d^{i+1}_{M^\bullet} \end{pmatrix} \\ &= \begin{pmatrix} -\mathrm{id}_{M^i} & 0 & 0 \\ 0 & 0 & 0 \\ -d^i_{M^\bullet} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d^i_{M^\bullet} & -\varphi^{i+1} & -\mathrm{id}_{M^{i+1}} \end{pmatrix} = \begin{pmatrix} -\mathrm{id}_{M^i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\varphi^{i+1} & -\mathrm{id}_{M^{i+1}} \end{pmatrix}. \end{aligned}$$

This proves that we have our desired homotopy equivalence. The square on the right commutes (even in $\mathsf{C}^*(\mathsf{A})$) by the very definition of the morphism $L[1]^\bullet \to \mathrm{MC}(\iota)^\bullet$. As for the one in the middle, we observe that the difference between the two morphisms $\mathrm{MC}(\varphi)^i \to \mathrm{MC}(\iota)^i$ is

$$\begin{pmatrix} \mathrm{id}_{M^i} & 0 \\ 0 & 0 \\ 0 & \varphi^{i+1} \end{pmatrix},$$

which is homotopic to zero via the homotopy $h^i : M^i \oplus L^{i+1} \to M^{i-1} \oplus L^i \oplus M^i$ given by

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\mathrm{id}_{M^i} & 0 \end{pmatrix}.$$

Indeed, the morphism $d^{i-1}_{MC(\iota)^\bullet} \circ h^i + h^{i+1} \circ d^i_{MC(\varphi)^\bullet}$ is represented by

$$\begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i & -\operatorname{id}_{M^i} \\ 0 & -d^i_{L^\bullet} & 0 \\ 0 & 0 & -d^i_{M^\bullet} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\operatorname{id}_{M^i} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\operatorname{id}_{M^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} d^i_{M^\bullet} & -\varphi^{i+1} \\ 0 & -d^{i+1}_{L^\bullet} \end{pmatrix}$$
$$= \begin{pmatrix} \operatorname{id}_{M^i} & 0 \\ 0 & 0 \\ d^i_{M^\bullet} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -d^i_{M^\bullet} & \varphi^{i+1} \end{pmatrix} = \begin{pmatrix} \operatorname{id}_{M^i} & 0 \\ 0 & 0 \\ 0 & \varphi^{i+1} \end{pmatrix},$$

which is equal to the difference calculated above. This completes the proof that the triangle on the right is distinguished if the left one is. For the converse, we observe that by applying what we just proved to the triangle on the right, supposed distinguished, five times, we arrive at the triangle

$$\begin{tikzcd}
& L[2]^\bullet \arrow[dr, "\overline{\varphi}[2]"] & \\
N[2]^\bullet \arrow[ur, "\overline{\rho}[2]", "+1"'] & & M[2]^\bullet, \arrow[ll, "\overline{\psi}[2]"]
\end{tikzcd}$$

which is distinguished if and only if the triangle on the left is. □

The next result shows that the mapping cone "almost" defines a functor from the category of morphisms (as seen in the proof of corollary 2.2.2) in $K^*(A)$ to $K^*(A)$ itself.

Lemma 2.4.3 Consider the following commutative diagram in $K^*(A)$ whose rows are distinguished triangles:

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\lambda}"] & M^\bullet \arrow[r, "\overline{\psi}"] \arrow[d, "\overline{\mu}"] & N^\bullet \arrow[r] & L[1]^\bullet \\
L'^\bullet \arrow[r, "\overline{\varphi}'"] & M'^\bullet \arrow[r, "\overline{\psi}'"] & N'^\bullet \arrow[r] & L'[1]^\bullet.
\end{tikzcd}$$

It exists a (not necessarily unique) morphism $\overline{\nu} : N^\bullet \to N'^\bullet$ making the diagram

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\lambda}"] & M^\bullet \arrow[r, "\overline{\psi}"] \arrow[d, "\overline{\mu}"] & N^\bullet \arrow[r] \arrow[d, dashed, "\overline{\nu}"] & L[1]^\bullet \arrow[d, "\overline{\lambda}[1]"] \\
L'^\bullet \arrow[r, "\overline{\varphi}'"] & M'^\bullet \arrow[r, "\overline{\psi}'"] & N'^\bullet \arrow[r] & L'[1]^\bullet
\end{tikzcd}$$

commute. That is, defining a morphism of triangles.

Proof. By composing with some isomorphisms, if necessary, we may assume that our original diagram is of the form

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{\varphi}"] \arrow[d, "\overline{\lambda}"] & M^\bullet \arrow[r] \arrow[d, "\overline{\mu}"] & \mathrm{MC}(\varphi)^\bullet \arrow[r] & L[1]^\bullet \\
L'^\bullet \arrow[r, "\overline{\varphi}'"] & M'^\bullet \arrow[r] & \mathrm{MC}(\varphi')^\bullet \arrow[r] & L'[1]^\bullet.
\end{tikzcd}$$

Since the square on the left commutes in $K^*(A)$, let $h^i : L^i \to M'^{i-1}$ be a collection of morphisms satisfying

$$\mu^i \circ \varphi^i - \varphi'^i \circ \lambda^i = d^{i-1}_{M'^\bullet} \circ h^i + h^{i+1} \circ d^i_{L^\bullet}$$

for all i. We then define our desired morphism $\nu^i : M^i \oplus L^{i+1} \to M'^i \oplus L'^{i+1}$ as

$$\begin{pmatrix} \mu^i & -h^{i+1} \\ 0 & \lambda^{i+1} \end{pmatrix}.$$

This is indeed a morphism of complexes since $\nu^i \circ d^{i-1}_{\mathrm{MC}(\varphi)^\bullet} - d^{i-1}_{\mathrm{MC}(\varphi')^\bullet} \circ \nu^{i-1}$ is represented by

$$\begin{aligned}
&\begin{pmatrix} \mu^i & -h^{i+1} \\ 0 & \lambda^{i+1} \end{pmatrix} \begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi'^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} \begin{pmatrix} \mu^{i-1} & -h^i \\ 0 & \lambda^i \end{pmatrix} \\
= &\begin{pmatrix} \mu^i \circ d^{i-1}_{M^\bullet} & -\mu^i \circ \varphi^i + h^{i+1} \circ d^i_{L^\bullet} \\ 0 & -\lambda^{i+1} \circ d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{M^\bullet} \circ \mu^{i-1} & -d^{i-1}_{M^\bullet} \circ h^i - \varphi'^i \circ \lambda^i \\ 0 & -d^i_{L^\bullet} \circ \lambda^i \end{pmatrix},
\end{aligned}$$

which is nothing but the zero matrix. This morphism makes the square on the middle commute due to the fact that

$$\begin{pmatrix} \mu^i & -h^{i+1} \\ 0 & \lambda^{i+1} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{M^i} \\ 0 \end{pmatrix} = \begin{pmatrix} \mu^i \\ 0 \end{pmatrix}$$

is equal to the composition of $\mu^i : M^i \to M'^i$ with the natural injection $M'^i \to M'^i \oplus L'^{i+1}$. Similarly, the square on the right commutes as

$$\begin{pmatrix} 0 & \mathrm{id}_{L'^{i+1}} \end{pmatrix} \begin{pmatrix} \mu^i & -h^{i+1} \\ 0 & \lambda^{i+1} \end{pmatrix} = \begin{pmatrix} 0 & \lambda^{i+1} \end{pmatrix}$$

coincides with the composition of the natural projection $M^i \oplus L^{i+1} \to L^{i+1}$ with $\lambda^{i+1} : L^{i+1} \to L'^{i+1}$. □

For an example of the lack of uniqueness, let $\overline{\nu} : M^\bullet \oplus L[1]^\bullet \to M^\bullet \oplus L[1]^\bullet$ be the morphism defined by

$$\begin{pmatrix} \overline{\mathrm{id}_{M^\bullet}} & \overline{\varphi} \\ 0 & \overline{\mathrm{id}_{L[1]^\bullet}} \end{pmatrix},$$

where $\varphi^\bullet$ is any morphism $L[1]^\bullet \to M^\bullet$. This morphism makes the diagram, whose rows are distinguished triangles,

$$\begin{tikzcd}
L^\bullet \arrow[r, "\overline{0}"] \arrow[d, equal] & M^\bullet \arrow[r] \arrow[d, equal] & M^\bullet \oplus L[1]^\bullet \arrow[r] \arrow[d, "\overline{\nu}"] & L[1]^\bullet \arrow[d, equal] \\
L^\bullet \arrow[r, "\overline{0}"] & M^\bullet \arrow[r] & M^\bullet \oplus L[1]^\bullet \arrow[r] & L[1]^\bullet
\end{tikzcd}$$

commute. This lack of uniqueness was the main motivation behind Grothendieck's unpublished 1991 manuscript *Les Dérivateurs*, which has almost 2000 pages.

Somewhat surprisingly, the lemmata that precedes amounts to essentially all the information needed to do homological algebra in $K^*(A)$. In our context, this was first formalized in *Jean-Louis Verdier*'s 1967 thesis as the notion of *triangulated category*, which we now present.

We begin with an additive category K endowed with an additive isomorphism of categories[5] $T : K \to K$ modeling the shift functor in $K^*(A)$. As before, a triangle in K is a diagram of the form

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L),$$

and a morphism of triangles is simply a commutative diagram

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & T(L) \arrow[d, "T(\lambda)"] \\
L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L').
\end{tikzcd}$$

We also specify a set of distinguished triangles that should satisfy the axioms below.[6]

(TR1) (a) Every triangle that is isomorphic to a distinguished triangle is also distinguished.

(b) For every morphism $\varphi : L \to M$ in K there is a distinguished triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L).$$

(c) For every object M the triangle

$$M \xrightarrow{\mathrm{id}_M} M \longrightarrow 0 \longrightarrow T(M)$$

is distinguished.

---

[5] In some texts the functor T is only required to be an equivalence of categories, instead of a genuine isomorphism. The resulting theory is more complicated as it is 2-categorical.

[6] Actually TR3 and half of TR2 follow from the rest of the axioms. The interested reader can check [36].

(TR2) A triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

is distinguished if and only if the triangle

$$M \xrightarrow{\psi} N \xrightarrow{\rho} T(L) \xrightarrow{-T(\varphi)} T(M)$$

is distinguished.

(TR3) Given a commutative diagram in K

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] & T(L) \\
L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L'),
\end{tikzcd}$$

whose rows are distinguished triangles, there's a morphism $\nu : N \to N'$ making the diagram

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & T(L) \arrow[d, "T(\lambda)"] \\
L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L')
\end{tikzcd}$$

commute.

(TR4) Suppose we are given these three distinguished triangles:

$$L \xrightarrow{\varphi} M \longrightarrow P \longrightarrow T(L),$$

$$M \xrightarrow{\psi} N \longrightarrow R \longrightarrow T(M),$$

$$L \xrightarrow{\psi\circ\varphi} N \longrightarrow Q \longrightarrow T(L).$$

Then there exists a distinguished triangle

$$P \longrightarrow Q \longrightarrow R \longrightarrow T(P)$$

making the diagram

$$\begin{tikzcd}
L \arrow[rr, bend left, "\psi\circ\varphi"] \arrow[dr, "\varphi"'] & & N \arrow[rr, bend left] \arrow[dr] & & R \arrow[rr, dashed, bend left] \arrow[dr] & & T(P) \\
& M \arrow[ur, "\psi"'] \arrow[dr] & & Q \arrow[ur, dashed] \arrow[dr] & & T(M) \arrow[ur] & \\
& & P \arrow[ur, dashed] \arrow[rr, bend right] & & T(L) \arrow[ur] & &
\end{tikzcd}$$

commute.

The object we're left with is a *triangulated category*.

Definition 2.4.2 — Triangulated category. A *triangulated category* is an additive category K, endowed with an additive automorphism $T : K \to K$ and a set of distinguished triangles satisfying the axioms TR1 to TR4 above.

By now, the reader probably wonders what is the axiom TR4 for. We affirm that it is a sort of palliative solution to the lack of uniqueness in the induced morphism of axiom TR3. Indeed, for every morphism $\varphi : L \to M$, the axiom TR1(b) gives an abstract mapping cone P defining a distinguished triangle

$$L \xrightarrow{\varphi} M \longrightarrow P \longrightarrow T(L).$$

Similarly, this axiom gives an abstract mapping cone R to a morphism $\psi : M \to N$ and an abstract mapping cone Q to the composition $\psi \circ \varphi : L \to N$. Naturally, we wonder how Q relates to P and R. The axiom TR4 affirms simply that they fit into a distinguished triangle

$$P \longrightarrow Q \longrightarrow R \longrightarrow T(P).$$

We leave a study of triangulated categories for the next section and end this one by proving that indeed $K^*(A)$ are triangulated categories. Once again, this isn't difficult at all, but there are a myriad of things that need to be verified.

Theorem 2.4.4 Let A be an additive category. Then the homotopic categories $K^*(A)$ are triangulated.

Proof. After our preliminary work, the only axiom that remains to be proven is the last one. For that we may suppose $P^\bullet = \mathrm{MC}(\varphi)^\bullet$, $R^\bullet = \mathrm{MC}(\psi)^\bullet$ and $Q^\bullet = \mathrm{MC}(\psi \circ \varphi)^\bullet$. We define morphisms $\alpha^i : P^i \to Q^i$ and $\beta^i : Q^i \to R^i$ as

$$\begin{pmatrix} \psi^i & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix},$$

respectively. The α^i define a morphism of complexes since $\alpha^i \circ d^{i-1}_{P^\bullet} - d^{i-1}_{Q^\bullet} \circ \alpha^{i-1}$ is represented by

$$\begin{aligned} &\begin{pmatrix} \psi^i & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} \begin{pmatrix} d^{i-1}_{M^\bullet} & -\varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{N^\bullet} & -\psi^i \circ \varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} \begin{pmatrix} \psi^{i-1} & 0 \\ 0 & \mathrm{id}_{L^i} \end{pmatrix} \\ &= \begin{pmatrix} \psi^i \circ d^{i-1}_{M^\bullet} & -\psi^i \circ \varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} - \begin{pmatrix} d^{i-1}_{N^\bullet} \circ \psi^{i-1} & -\psi^i \circ \varphi^i \\ 0 & -d^i_{L^\bullet} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Similarly, the β^i define a morphism of complexes since $\beta^i \circ d_{Q^\bullet}^{i-1} - d_{R^\bullet}^{i-1} \circ \beta^{i-1}$ is represented by

$$\begin{aligned}
&\begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix} \begin{pmatrix} d_{N^\bullet}^{i-1} & -\psi^i \circ \varphi^i \\ 0 & -d_{L^\bullet}^i \end{pmatrix} - \begin{pmatrix} d_{N^\bullet}^{i-1} & -\psi^i \\ 0 & -d_{M^\bullet}^i \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^{i-1}} & 0 \\ 0 & \varphi^i \end{pmatrix} \\
&= \begin{pmatrix} d_{N^\bullet}^{i-1} & -\psi^i \circ \varphi^i \\ 0 & -\varphi^{i+1} \circ d_{L^\bullet}^i \end{pmatrix} - \begin{pmatrix} d_{N^\bullet}^{i-1} & -\psi^i \circ \varphi^i \\ 0 & -d_{M^\bullet}^i \circ \varphi^i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

We also define a morphism $\gamma^\bullet : R^\bullet \to P[1]^\bullet$ as the composition $R^\bullet \to M[1]^\bullet \to P[1]^\bullet$. We must now verify that

$$P^\bullet \xrightarrow{\alpha^\bullet} Q^\bullet \xrightarrow{\beta^\bullet} R^\bullet \xrightarrow{\gamma^\bullet} P[1]^\bullet.$$

is a distinguished triangle and that those morphisms fit into the commutative diagram of the axiom TR4. For clarity, we number the relevant parts of this diagram and rewrite it here.

$$\begin{tikzcd}
L^\bullet \arrow[rr, bend left, "\psi^\bullet \circ \varphi^\bullet"] \arrow[dr, "\varphi^\bullet"'] & (1) & N^\bullet \arrow[rr, bend left] \arrow[dr] & (3) & R^\bullet \arrow[rr, bend left, "\gamma^\bullet"] \arrow[dr] & (6) & P[1]^\bullet \\
& M^\bullet \arrow[ur, "\psi^\bullet"'] \arrow[dr] & (2) & Q^\bullet \arrow[ur, "\beta^\bullet"'] \arrow[dr] & (5) & M[1]^\bullet \arrow[ur] & \\
& & P^\bullet \arrow[ur, "\alpha^\bullet"'] \arrow[rr, bend right] & (4) & L[1]^\bullet \arrow[ur] & &
\end{tikzcd}$$

The triangles (1) and (6) commute by the very definition of the morphisms involved. The square (2) commutes since

$$\begin{pmatrix} \mathrm{id}_{N^i} \\ 0 \end{pmatrix} \psi^i = \begin{pmatrix} \psi^i & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{M^i} \\ 0 \end{pmatrix}.$$

The triangle (3) commutes since

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{N^i} \\ 0 \end{pmatrix}.$$

The triangle (4) commutes since

$$\begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} \begin{pmatrix} \psi^i & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix}.$$

Finally, the square (5) commutes since

$$\begin{pmatrix} 0 & \mathrm{id}_{M^{i+1}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix} = \varphi^{i+1} \begin{pmatrix} 0 & \mathrm{id}_{L^{i+1}} \end{pmatrix}.$$

In order to show that the triangle we defined is distinguished we'll define morphisms $\rho^\bullet : \mathrm{MC}(\alpha)^\bullet \to R^\bullet$ and $\sigma^\bullet : R^\bullet \to \mathrm{MC}(\alpha)^\bullet$ determining an isomorphism of triangles

$$\begin{tikzcd}
P^\bullet \arrow[r, "\alpha^\bullet"] \arrow[d, equal] & Q^\bullet \arrow[r, "\beta^\bullet"] \arrow[d, equal] & R^\bullet \arrow[r, "\gamma^\bullet"] \arrow[d, shift left, "\sigma^\bullet"] & P[1]^\bullet \arrow[d, equal] \\
P^\bullet \arrow[r, "\alpha^\bullet"] & Q^\bullet \arrow[r] & \mathrm{MC}(\alpha)^\bullet \arrow[r] \arrow[u, shift left, "\rho^\bullet"] & P[1]^\bullet .
\end{tikzcd}$$

The morphisms $\rho^i : N^i \oplus L^{i+1} \oplus M^{i+1} \oplus L^{i+2} \to N^i \oplus M^{i+1}$ and $\sigma^i : N^i \oplus M^{i+1} \to N^i \oplus L^{i+1} \oplus M^{i+1} \oplus L^{i+2}$ are defined as

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix},$$

respectively. They define morphisms of complexes since $\rho^{i+1} \circ d^i_{\mathrm{MC}(\alpha)^\bullet} - d^i_{R^\bullet} \circ \rho^i$ is represented by

$$\begin{pmatrix} \mathrm{id}_{N^{i+1}} & 0 & 0 & 0 \\ 0 & \varphi^{i+2} & \mathrm{id}_{M^{i+2}} & 0 \end{pmatrix} \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -d^{i+1}_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+2}} \\ 0 & 0 & -d^{i+1}_{M^\bullet} & \varphi^{i+2} \\ 0 & 0 & 0 & d^{i+2}_{L^\bullet} \end{pmatrix}$$
$$- \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \\ 0 & -d^{i+1}_{M^\bullet} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} =$$
$$\begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -\varphi^{i+2} \circ d^{i+1}_{L^\bullet} & -d^{i+1}_{M^\bullet} & 0 \end{pmatrix} - \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -d^{i+1}_{M^\bullet} \circ \varphi^{i+1} & -d^{i+1}_{M^\bullet} & 0 \end{pmatrix}$$

and $\sigma^{i+1} \circ d^i_{R^\bullet} - d^i_{MC(\alpha)^\bullet} \circ \sigma^i$ is represented by

$$\begin{pmatrix} \mathrm{id}_{N^{i+1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \\ 0 & -d^{i+1}_{M^\bullet} \end{pmatrix} -$$

$$\begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -d^{i+1}_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+2}} \\ 0 & 0 & -d^{i+1}_{M^\bullet} & \varphi^{i+2} \\ 0 & 0 & 0 & d^{i+2}_{L^\bullet} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \\ 0 & 0 \\ 0 & -d^{i+1}_{M^\bullet} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} d^i_{N^\bullet} & -\psi^{i+1} \\ 0 & 0 \\ 0 & -d^{i+1}_{M^\bullet} \\ 0 & 0 \end{pmatrix}.$$

In both cases the result is the zero matrix. We now affirm that the morphisms $\rho^\bullet$ and $\sigma^\bullet$ define a homotopy equivalence. The composition $\rho^\bullet \circ \sigma^\bullet$ is equal to the identity morphism on $R^\bullet$ as

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \end{pmatrix},$$

and the morphism $\sigma^\bullet \circ \rho^\bullet - \mathrm{id}_{MC(\alpha)^\bullet}$, represented by

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} - \begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+1}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{M^{i+1}} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{L^{i+2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathrm{id}_{L^{i+1}} & 0 & 0 \\ 0 & \varphi^{i+1} & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{id}_{L^{i+2}} \end{pmatrix},$$

is homotopic to zero via the homotopy $h^i : N^i \oplus L^{i+1} \oplus M^{i+1} \oplus L^{i+2} \to N^{i-1} \oplus L^i \oplus M^i \oplus L^{i+1}$ given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+1}} & 0 & 0 \end{pmatrix}.$$

Indeed, the composition $d^{i-1}_{\mathrm{MC}(\alpha)^\bullet} \circ h^i + h^{i+1} \circ d^i_{\mathrm{MC}(\alpha)^\bullet}$ is represented by the matrix

$$\begin{pmatrix} d^{i-1}_{N^\bullet} & -\psi^i \circ \varphi^i & -\psi^i & 0 \\ 0 & -d^i_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+1}} \\ 0 & 0 & -d^i_{M^\bullet} & \varphi^{i+1} \\ 0 & 0 & 0 & d^{i+1}_{L^\bullet} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+1}} & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_{L^{i+2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} d^{i}_{N^\bullet} & -\psi^{i+1} \circ \varphi^{i+1} & -\psi^{i+1} & 0 \\ 0 & -d^{i+1}_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+2}} \\ 0 & 0 & -d^{i+1}_{M^\bullet} & \varphi^{i+2} \\ 0 & 0 & 0 & d^{i+2}_{L^\bullet} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathrm{id}_{L^{i+1}} & 0 & 0 \\ 0 & \varphi^{i+1} & 0 & 0 \\ 0 & d^{i+1}_{L^\bullet} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -d^{i+1}_{L^\bullet} & 0 & -\mathrm{id}_{L^{i+2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\mathrm{id}_{L^{i+1}} & 0 & 0 \\ 0 & \varphi^{i+1} & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{id}_{L^{i+2}} \end{pmatrix},$$

which coincides with the one representing $\sigma^\bullet \circ \rho^\bullet - \mathrm{id}_{\mathrm{MC}(\alpha)^\bullet}$. It remains only to show that $\rho^\bullet$ defines a morphism of triangles. That is, that the associated diagram commutes. The composition of the natural injection $Q^\bullet \to \mathrm{MC}(\alpha)^\bullet$ with $\rho^\bullet$ is given by

$$\begin{pmatrix} \mathrm{id}_{N^i} & 0 & 0 & 0 \\ 0 & \varphi^{i+1} & \mathrm{id}_{M^{i+1}} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \mathrm{id}_{L^{i+1}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & \varphi^{i+1} \end{pmatrix},$$

which is nothing but β^i. Since $\sigma^\bullet$ is the inverse of $\rho^\bullet$ in $\mathsf{K}^*(\mathsf{A})$, it suffices to show that the composition of $\sigma^\bullet$ with the natural projection $\mathrm{MC}(\alpha)^\bullet \to \mathsf{P}[1]^\bullet$ is $\gamma^\bullet$. This holds since

$$\begin{pmatrix} 0 & 0 & \mathrm{id}_{M^{i+1}} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{L^{i+2}} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{N^i} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_{M^{i+1}} \\ 0 & 0 \end{pmatrix},$$

which is equal to γ^i. The proof is at long last over. □

2.5 Triangulated categories

After proving that the homotopic categories are triangulated in the last section, we now delve into the world of triangulated categories. The formal results we'll obtain will not only be valid and useful for the homotopic categories, but also for the derived category in the next chapter.

We begin by understanding what are the natural functors between triangulated categories, preserving their extra structure.

Definition 2.5.1 — Triangulated functor. Let (K, T) and (K', T') be triangulated categories. A *triangulated functor* from K to K' is an additive functor $F : K \to K'$, together with a natural isomorphism $\tau : F \circ T \to T' \circ F$, such that for every distinguished triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L),$$

in K, the triangle

$$F(L) \xrightarrow{F(\varphi)} F(M) \xrightarrow{F(\psi)} F(N) \xrightarrow{\tau_L \circ F(\rho)} T'(F(L))$$

is distinguished in K'.

Whenever we say that two triangulated categories are equivalent, it is to be understood that the functor defining the equivalence of categories is triangulated. Also, if $F : A \to B$ is an additive functor between additive categories, then the induced functor $F : K(A) \to K(B)$ is triangulated. Indeed, an additive functor commutes both with mapping cones and with the shift functor.

Recall that, given a morphism $\varphi : L \to M$ in a triangulated category K, the axiom TR1 gives a distinguished triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L).$$

As in the homotopy category, we say that N is the cone of φ. We'll soon see that it is unique up to isomorphism. For now, this will allow us to define the natural notion of a triangulated (full) subcategory.

Definition 2.5.2 — Triangulated subcategory. Let (K, T) be a triangulated category. A *triangulated subcategory* of K is a full additive subcategory $C \subset K$, which is closed under cones and under the action of T. That is, the cone of a morphism in C is in C and $T(L) \in C$ whenever $L \in C$.

Surely, if C is a triangulated subcategory of (K, T), the restriction of T to C and the collection of distinguished triangles in K whose objects are in C gives a structure of triangulated category to C. Moreover, the inclusion functor $C \to K$ is triangulated.

As we observed before, the long sequence induced by the mapping cone of a morphism is a complex in the homotopic category (but not in the category of complexes). This generalizes to triangulated categories. In particular, it follows that the category of complexes cannot be triangulated (with respect to the usual shift functor and mapping cones).

Proposition 2.5.1 Let K be a triangulated category and

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

be a distinguished triangle. Then the compositions $\psi \circ \varphi$, $\rho \circ \psi$ and $T(\varphi) \circ \rho$ are zero.

Proof. The axiom TR1 says that the cone of the identity morphism id_L is zero. So, by the axiom TR3, we have a dashed arrow making the diagram

$$\begin{tikzcd}
L \arrow[r, "\mathrm{id}_L"] \arrow[d, "\mathrm{id}_L"] & L \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d, dashed] & T(L) \arrow[d, "T(\mathrm{id}_L)"] \\
L \arrow[r, "\varphi"'] & M \arrow[r, "\psi"'] & N \arrow[r, "\rho"'] & T(L)
\end{tikzcd}$$

commute. This proves that $\psi \circ \varphi = 0$. Now, the axiom TR2 says that the triangles

$$M \xrightarrow{\psi} N \xrightarrow{\rho} T(L) \xrightarrow{-T(\varphi)} T(M)$$

and

$$N \xrightarrow{\rho} T(L) \xrightarrow{-T(\varphi)} T(M) \xrightarrow{-T(\psi)} T^2(N)$$

are distinguished. So, by applying what we just proved to these triangles, we obtain $\rho \circ \psi = 0$ and $T(\varphi) \circ \rho = 0$. □

Duality arguments abound in category theory, as we clearly saw in the chapter about abelian categories. In order to use such arguments in our present context, we need to know that the opposite of a triangulated category is also triangulated.

Proposition 2.5.2 Let K be a triangulated category and let $D : K \to K^{op}$ be the contravariant functor sending each object to itself and inverting all the arrows. We define an additive isomorphism of categories $T^{op} : K^{op} \to K^{op}$ as $D \circ T^{-1} \circ D^{-1}$ and we say that a triangle of K^{op} is distinguished if it is of the form

$$N \xrightarrow{D(\psi)} M \xrightarrow{D(\varphi)} L \xrightarrow{D(-T^{-1}(\rho))} T^{op}(N),$$

where

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

is a distinguished triangle in K. Then K^{op} is a triangulated category.

Considering that the proof of this result amounts only to a formal verification of the axioms, and that it won't add new useful ideas or techniques to the arsenal of the reader, we won't write it here. In case the reader wants to see it anyway, a full proof is available online on [37].

We also remark that the collection of distinguished triangles in K^{op} is motivated by the fact that the axiom TR2 says that a triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)$$

is distinguished if and only if its "reverse rotation"

$$T^{-1}(N) \xrightarrow{-T^{-1}(\rho)} L \xrightarrow{\varphi} M \xrightarrow{\psi} N$$

is. By inverting the triangle above, we obtain a distinguished triangle in the opposite category.

In the lingo of triangulated categories, the content of the proposition 2.2.3 is that the functor $H^\bullet : K^*(A) \to C^*(A)$ sends distinguished triangles to exact triangles. We axiomatize this behavior.

Definition 2.5.3 Let K be a triangulated category and A be an abelian category. We say that an additive functor $H : K \to A$ is *cohomological* is, for every distinguished triangle

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L),$$

the sequence

$$H(L) \xrightarrow{H(\varphi)} H(M) \xrightarrow{H(\psi)} H(N)$$

is exact in A.

Since we can use the axiom TR2 to rotate our distinguished triangles, we obtain a (infinite) sequence of distinguished triangles

$$\begin{aligned}
&L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L)\\
&M \xrightarrow{\psi} N \xrightarrow{\rho} T(L) \xrightarrow{-T(\varphi)} T(M)\\
&N \xrightarrow{\rho} T(L) \xrightarrow{-T(\varphi)} T(M) \xrightarrow{-T(\psi)} T(N)\\
&T(L) \xrightarrow{-T(\varphi)} T(M) \xrightarrow{-T(\psi)} T(N) \xrightarrow{-T(\rho)} T^2(L).
\end{aligned}$$

Moreover, we can make sure that in each triangle the first two morphisms don't have a minus sign. For example, the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "\rho"] \arrow[d, "\mathrm{id}_N"] & T(L) \arrow[r, "-T(\varphi)"] \arrow[d, "\mathrm{id}_{T(L)}"] & T(M) \arrow[r, "-T(\psi)"] \arrow[d, "-\mathrm{id}_{T(M)}"] & T(N) \arrow[d, "T(\mathrm{id}_N)"] \\
N \arrow[r, "\rho"'] & T(L) \arrow[r, "T(\varphi)"'] & T(M) \arrow[r, "T(\psi)"'] & T(N)
\end{tikzcd}$$

shows that the third triangle is isomorphic to a triangle with the same objects but whose first two morphisms "don't have a minus sign". By applying a cohomological functor H, we obtain a long exact sequence associated with our original distinguished triangle

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & H(L) \arrow[r, "H(\varphi)"] & H(M) \arrow[r, "H(\psi)"] & H(N) \arrow[dll, rounded corners, "H(\rho)"'] & \\
 & H(T(L)) \arrow[r, "H(T(\varphi))"] & H(T(M)) \arrow[r, "H(T(\psi))"] & H(T(N)) \arrow[r, dashed] & \cdots .
\end{tikzcd}$$

As we just hinted, the functor $H^i : K^*(A) \to A$, for all i, is cohomological. But it isn't by all means the only one. The proposition below gives two other cohomological functors which will allow the use of the Yoneda lemma to study triangulated categories.

Proposition 2.5.3 Let K be a triangulated category. Then, the functors

$$\mathrm{Hom}_K(P, -) : K \to \mathsf{Ab} \qquad \text{and} \qquad \mathrm{Hom}_K(-, P) : K^{op} \to \mathsf{Ab},$$

for every object P of K, are cohomological.

Proof. We'll only prove the covariant statement, for $\mathrm{Hom}_K(-, P) = \mathrm{Hom}_{K^{op}}(P, -)$ implies the other. Consider the following distinguished triangle in K:

$$L \xrightarrow{\varphi} M \xrightarrow{\psi} N \xrightarrow{\rho} T(L).$$

In order to show that $\mathrm{Hom}_K(P, -)$ is cohomological, we need to prove that the induced sequence

$$\mathrm{Hom}_K(P, L) \longrightarrow \mathrm{Hom}_K(P, M) \longrightarrow \mathrm{Hom}_K(P, N)$$

is exact. Since $\psi \circ \varphi = 0$, due to the proposition 2.5.1, it suffices to show that for every $\alpha : P \to M$ such that $\psi \circ \alpha = 0$, there exists a morphism $\beta : P \to L$ such that $\alpha = \varphi \circ \beta$.

Now, consider the diagram below:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & T(P) \arrow[r, "-\mathrm{id}_{T(P)}"] & T(P) \arrow[d, "T(\alpha)"] \\
M \arrow[r, "\psi"'] & N \arrow[r, "\rho"'] & T(L) \arrow[r, "-T(\varphi)"'] & T(M).
\end{tikzcd}$$

Its lower row is a distinguished triangle, since it is nothing but our original triangle rotated with help of the axiom TR2. The upper row is also a distinguished triangle by the axioms TR1 and TR2. The axiom TR3 gives a morphism $T(P) \to T(L)$ making it commute which, since T is fully-faithful, is of the form $T(\beta)$ for exactly one $\beta : P \to L$. Since the square on the right commutes, $T(\alpha) = T(\varphi) \circ T(\beta) = T(\varphi \circ \beta)$. This implies that $\alpha = \varphi \circ \beta$ and finishes the proof. □

We now prove a couple of interesting corollaries. The one below is a form of the five lemma for triangulated categories.

Corollary 2.5.4 Consider the following morphism of distinguished triangles in a triangulated category K:

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\lambda"] & M \arrow[r] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & T(L) \arrow[d, "T(\lambda)"] \\
L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & T(L').
\end{tikzcd}$$

If two of the vertical morphisms λ, μ and ν are isomorphism, then so is the third.

Proof. Without loss of generality, we may suppose that λ and μ are isomorphisms. Let P be an object of K and $H := \mathrm{Hom}_{\mathsf{K}}(P, -)$. By applying H, we get a commutative diagram of abelian groups

$$\begin{tikzcd}
H(L) \arrow[r] \arrow[d, "H(\lambda)"] & H(M) \arrow[r] \arrow[d, "H(\mu)"] & H(N) \arrow[r] \arrow[d, "H(\nu)"] & H(T(L)) \arrow[r] \arrow[d, "H(T(\lambda))"] & H(T(M)) \arrow[d, "H(T(\mu))"] \\
H(L') \arrow[r] & H(M') \arrow[r] & H(N') \arrow[r] & H(T(L')) \arrow[r] & H(T(M'))
\end{tikzcd}$$

which, due to the proposition above and its preceding discussion, has exact rows. The five lemma (proposition 1.6.2) then implies that $F(\nu)$ is an isomorphism of abelian groups and, in particular, of sets. Since this holds for every P, the Yoneda lemma implies that ν is an isomorphism. ☐

If the reader prefers to avoid the Yoneda lemma, we can arrive at the same conclusion in a direct way. Since

$$\begin{aligned} H(\nu) : \mathrm{Hom}_{\mathsf{K}}(P, N) &\to \mathrm{Hom}_{\mathsf{K}}(P, N') \\ \alpha &\mapsto \nu \circ \alpha \end{aligned}$$

is an isomorphism for all P, we can take $P = N'$ and conclude that there is some $\alpha : N' \to N$ such that $\nu \circ \alpha = \mathrm{id}_{N'}$. That is, ν has a right inverse. The same argument with the contravariant hom functor gives a left inverse to ν, proving that it is an isomorphism.

One corollary of the result above is that the cone of a morphism is unique up to isomorphism.

Corollary 2.5.5 Let K be a triangulated category and $\varphi : L \to M$ be a morphism in K. Then the cone N of φ is unique up to isomorphism.

Proof. Suppose that N' is another cone of φ. The axiom TR3 gives a morphism $\nu : N \to N'$ making the diagram

$$\begin{tikzcd}
L \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_L"] & M \arrow[r, "\psi"] \arrow[d, "\mathrm{id}_M"] & N \arrow[r, "\rho"] \arrow[d, dashed, "\nu"] & T(L) \arrow[d, "T(\mathrm{id}_L)"] \\
L \arrow[r, "\varphi"'] & M \arrow[r] & N' \arrow[r] & T(L)
\end{tikzcd}$$

commute. The preceding corollary then implies that ν is an isomorphism. □

We observe that the non-uniqueness in the induced morphism of the axiom TR3 implies that the isomorphism ν above is not necessarily unique. In particular, the cone of φ is not functorial in φ. As discussed right after the definition 2.4.2, this is the *raison d'être* of the axiom TR4.

Corollary 2.5.6 Let K be a triangulated category and

$$\begin{tikzcd}
L \arrow[r, "\varphi"] & M \arrow[r, "\psi"] & N \arrow[r, "\rho"] & T(L)
\end{tikzcd}$$

be a distinguished triangle. Then φ is an isomorphism if and only if N is isomorphic to the zero object.

Proof. Suppose that φ is an isomorphism, and let $\varphi^{-1} : M \to L$ be its inverse. Since two of the vertical morphisms in the diagram

$$\begin{tikzcd}
L \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_L"] & M \arrow[r, "\psi"] \arrow[d, "\varphi^{-1}"] & N \arrow[r, "\rho"] \arrow[d] & T(L) \arrow[d, "T(\mathrm{id}_L)"] \\
L \arrow[r, "\mathrm{id}_L"'] & L \arrow[r] & 0 \arrow[r] & T(L)
\end{tikzcd}$$

are isomorphisms, so is $N \to 0$. Conversely, suppose that N is isomorphic to zero. By rotating backwards our distinguished triangle, we obtain the diagram below

$$\begin{tikzcd}
T^{-1}(N) \arrow[r, "-T^{-1}(\rho)"] \arrow[d] & L \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_L"] & M \arrow[r, "\psi"] \arrow[d, dashed] & N \arrow[d] \\
0 \arrow[r] & L \arrow[r, "\mathrm{id}_L"'] & L \arrow[r] & 0,
\end{tikzcd}$$

whose rows are distinguished triangles. The dashed morphism, induced by the axiom TR3, is an isomorphism by the proposition above. The commutativity of the diagram then implies that so is φ. □

We're now in position to explain why the homotopic category (and the derived category) are usually not abelian. The reader may remember the next result as saying that "in a triangulated category, monomorphisms and epimorphisms split".

Proposition 2.5.7 Let K be a triangulated category. If $\varphi : L \to M$ is a monomorphism, then there exists $\rho : M \to L$ such that $\rho \circ \varphi = \mathrm{id}_L$. Dually, if $\psi : M \to N$ is an epimorphism, then there exists $\sigma : N \to M$ such that $\psi \circ \sigma = \mathrm{id}_N$.

Proof. Suppose that $\varphi : L \to M$ is a monomorphism. By the axioms TR1(b) and TR2, there exists a distinguished triangle of the form

$$T^{-1}(N) \longrightarrow L \xrightarrow{\varphi} M \longrightarrow N.$$

Due to the proposition 2.5.1, the composition $T^{-1}(N) \to L \to M$ is zero. But, since φ is a monomorphism, it follows that $T^{-1}(N) \to L$ is also zero. As $\mathrm{Hom}_{\mathsf{K}}(-, L)$ is cohomological, we get an exact sequence

$$\mathrm{Hom}_{\mathsf{K}}(M, L) \longrightarrow \mathrm{Hom}_{\mathsf{K}}(L, L) \xrightarrow{0} \mathrm{Hom}_{\mathsf{K}}(T^{-1}(N), L),$$

which implies that $\mathrm{Hom}_{\mathsf{K}}(M, L) \to \mathrm{Hom}_{\mathsf{K}}(L, L)$ is surjective. In particular, there exists $\rho : M \to L$ such that $\rho \circ \varphi = \mathrm{id}_L$. The other statement follows by duality. ☐

The proposition above says, in particular, that if $\mathsf{K}(\mathsf{A})$ is abelian, then every exact sequence splits, due to the splitting lemma (theorem 1.4.1). In fact, this also implies that every exact sequence in A splits.

Corollary 2.5.8 Let A be an abelian category and suppose that $\mathsf{K}(\mathsf{A})$ is abelian. Then every exact sequence in A splits.

Proof. Let $\varphi : A \to B$ be a monomorphism in A and see this morphism in $\mathsf{K}(\mathsf{A})$. Since we suppose that the homotopy category is abelian, we can factor φ as $\mathrm{im}\,\varphi \circ \mathrm{coim}\,\varphi$ in $\mathsf{K}(\mathsf{A})$. As $\mathrm{im}\,\varphi$ is a monomorphism and $\mathrm{coim}\,\varphi$ is an epimorphism, the preceding proposition gives morphisms ρ and σ such that $\rho \circ \mathrm{im}\,\varphi = \mathrm{id}$ and $(\mathrm{coim}\,\varphi) \circ \sigma = \mathrm{id}$.

Let $\alpha = \sigma \circ \rho : B \to A$. Observe that α is in A, since A embeds fully faithfully in $\mathsf{K}(\mathsf{A})$, and that

$$\begin{aligned} \varphi \circ \alpha \circ \varphi &= (\mathrm{im}\,\varphi \circ \mathrm{coim}\,\varphi) \circ (\sigma \circ \rho) \circ (\mathrm{im}\,\varphi \circ \mathrm{coim}\,\varphi) \\ &= \mathrm{im}\,\varphi \circ \underbrace{(\mathrm{coim}\,\varphi \circ \sigma)}_{\mathrm{id}} \circ \underbrace{(\rho \circ \mathrm{im}\,\varphi)}_{\mathrm{id}} \circ \mathrm{coim}\,\varphi \\ &= \mathrm{im}\,\varphi \circ \mathrm{coim}\,\varphi = \varphi. \end{aligned}$$

But φ is a monomorphism in A and so $\alpha \circ \varphi = \mathrm{id}_A$. The result then follows by the splitting lemma. ☐

This result was proved only for the homotopy category, since we are yet to see the formal definition of the derived category. But the reader will realize in due time that the same argument also proves that if $\mathsf{D}(\mathsf{A})$ is abelian, then every exact sequence in A splits.[7]

---

[7] An abelian category where every exact sequence splits is said to be *semisimple*.

3 The derived category

As hinted in the previous chapter, our goal is to eventually study the derived category $\mathsf{D}(\mathsf{A})$, which will be constructed from the homotopy category $\mathsf{K}(\mathsf{A})$ by inverting all the quasi-isomorphisms. Unlike homotopy equivalences, quasi-isomorphisms does not define an equivalence relation, precluding us from defining $\mathsf{D}(\mathsf{A})$ by quotienting the hom-sets as we did in the homotopy category. We shall need more powerful machinery; the localization of categories.

3.1 Localization of categories

The main idea of this section is very simple: given a category C and a collection of morphisms S in C, we will define a category $S^{-1}\mathsf{C}$, along with a functor $Q : \mathsf{C} \to S^{-1}\mathsf{C}$ sending all elements of S to isomorphisms in $S^{-1}\mathsf{C}$, and such that Q is universal with this property. In other words, we'll establish the following theorem.

Theorem 3.1.1 Let C be a category and S a collection of morphisms in C. Then there exists a category $S^{-1}\mathsf{C}$ and a functor $Q : \mathsf{C} \to S^{-1}\mathsf{C}$ satisfying the following properties:

(a) for every $s \in S$, $Q(s)$ is an isomorphism in $S^{-1}\mathsf{C}$;

(b) if $F : \mathsf{C} \to \mathsf{D}$ is a functor such that $F(s)$ is an isomorphism for every $s \in S$, there exists a unique functor $S^{-1}\mathsf{C} \to \mathsf{D}$ making the diagram

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathsf{D} \\
S^{-1}\mathsf{C} \arrow[ur, dashed] &
\end{tikzcd}$$

commute.

Moreover, $S^{-1}\mathsf{C}$ is unique up to a unique isomorphism.

We say that $S^{-1}\mathsf{C}$ is the *localization* of C with respect to S. Before going on to the proof of this result, it is useful to understand how the explicit construction of $S^{-1}\mathsf{C}$ works. Let's begin by posing that $S^{-1}\mathsf{C}$ should have the same objects as C. As for the morphisms, if M and N are objects of C, we define a *path* from M to N to be a diagram of the form

$$M \xrightarrow{f_0} L_1 \xleftarrow{s_1} L_2 \xleftarrow{s_2} \cdots \xrightarrow{f_{n-1}} L_n \xleftarrow{s_n} N,$$

where $L_1, \dots, L_n$ are objects of C, the arrows f_i to the right are morphisms of C, and the arrows s_i to the left are elements of S. We denote such a path symbolically as $s_n^{-1} \circ f_{n-1} \circ \cdots \circ s_2^{-1} \circ s_1^{-1} \circ f_0$. Now, in order for this representation to function as it should, we define an equivalence relation on paths by imposing that compositions behave well

$$L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1} \quad \text{is equivalent to} \quad L_{i-1} \xrightarrow{f_i \circ f_{i-1}} L_{i+1},$$

that we may ignore identities

$$L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{\mathrm{id}_{L_i}} L_i \xrightarrow{f_i} L_{i+1} \quad \text{is equivalent to} \quad L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1},$$

and that arrows to the left correspond to inverses

$$\begin{array}{ccc} M \xrightarrow{s} N \xleftarrow{s} M & & M \xrightarrow{\mathrm{id}_M} M \\ & \text{are equivalent to} & \\ N \xleftarrow{s} M \xrightarrow{s} N & & N \xrightarrow{\mathrm{id}_N} N. \end{array}$$

We then define a morphism $M \to N$ in the localization $S^{-1}\mathsf{C}$ to be an equivalence class of paths from M to N. Composition of morphisms is given simply by concatenation. Moreover, the identity morphism id_M in $S^{-1}\mathsf{C}$ of an object M is the equivalence class of the path

$$M \xrightarrow{\mathrm{id}_M} M.$$

Finally, the functor $Q : \mathsf{C} \to S^{-1}\mathsf{C}$ is given by the identity on objects and sends a morphism $f : M \to N$ to the equivalence class of the path

$$M \xrightarrow{f} N.$$

We now verify all the formal details for the proof of our theorem.

Proof of theorem 3.1.1. First and foremost, we remark that we have indeed defined an equivalence relation on paths and that $S^{-1}\mathsf{C}$ is indeed a category. Also, the image $Q(s)$ of any morphism $s : M \to N$ in S is indeed an isomorphism in $S^{-1}\mathsf{C}$, whose inverse is represented by

$$N \xleftarrow{s} M.$$

As for the universal property, let $F : \mathsf{C} \to \mathsf{D}$ be a functor such that $F(s)$ is an isomorphism for every $s \in S$. We define a functor $G : S^{-1}\mathsf{C} \to \mathsf{D}$ which is equal to F on objects and sends the equivalence class of a path

$$M \xrightarrow{f_0} L_1 \xleftarrow{s_1} L_2 \xleftarrow{s_2} \cdots \xrightarrow{f_{n-1}} L_n \xleftarrow{s_n} N$$

to the composition

$$\mathsf{F}(M) \xrightarrow{\mathsf{F}(f_0)} \mathsf{F}(L_1) \xrightarrow{\mathsf{F}(s_1)^{-1}} \mathsf{F}(L_2) \xrightarrow{\mathsf{F}(s_2)^{-1}} \cdots \xrightarrow{\mathsf{F}(f_{n-1})} \mathsf{F}(L_n) \xrightarrow{\mathsf{F}(s_n)^{-1}} \mathsf{F}(N).$$

Since functors preserve composition and identities, this is independent of the choice of representative. This functor indeed satisfies $F = G \circ Q$ and, by construction, is uniquely determined by F. The uniqueness of the localization follows as usual from universal properties. □

As a quick corollary, we observe that localization behaves well with relation to the opposite category.

> **Corollary 3.1.2** Let C be a category and S a collection of morphisms in C. The category $(S^{-1}\mathsf{C})^{\text{op}}$ is isomorphic to the localization of C^{op} with respect to S^{op}.

Proof. Consider the functor $Q^{\text{op}} : \mathsf{C}^{\text{op}} \to (S^{-1}\mathsf{C})^{\text{op}}$. It is clear that Q^{op} sends elements of S^{op} to isomorphisms. Now, if $F : \mathsf{C}^{\text{op}} \to \mathsf{D}$ is a functor sending elements of S^{op} to isomorphisms, its opposite $F^{\text{op}} : \mathsf{C} \to \mathsf{D}^{\text{op}}$ sends elements of S to isomorphisms and so factors through the localization $S^{-1}\mathsf{C}$:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "F^{\text{op}}"] \arrow[d, "Q"'] & \mathsf{D}^{\text{op}} \\
S^{-1}\mathsf{C}. \arrow[ur, dashed] &
\end{tikzcd}$$

The image of the diagram above by the opposite category functor gives the existence of a unique functor $(S^{-1}\mathsf{C})^{\text{op}} \to \mathsf{D}$ making the diagram

$$\begin{tikzcd}
\mathsf{C}^{\text{op}} \arrow[r, "F"] \arrow[d, "Q^{\text{op}}"'] & \mathsf{D} \\
(S^{-1}\mathsf{C})^{\text{op}} \arrow[ur, dashed] &
\end{tikzcd}$$

commute. The uniqueness of the localization then yields the desired result. □

The homotopy category $\mathsf{K}(\mathsf{A})$ is already the localization of $\mathsf{C}(\mathsf{A})$ with respect to the collection of homotopy equivalences. In addition, we'll define the derived category $\mathsf{D}(\mathsf{A})$ as the localization of $\mathsf{C}(\mathsf{A})$ (or, as we've seen, $\mathsf{K}(\mathsf{A})$) with respect to the collection of quasi-isomorphisms. Before going any further, let's check another interesting example.

■ **Example 3.1.1 — Lie's third theorem.** Let LieGrp be the category of *connected* Lie groups and LieAlg be the category of finite-dimensional Lie algebras. The *tangent space at the identity* functor

$$\mathsf{LieGrp} \to \mathsf{LieAlg}$$

is faithful and essentially surjective, but it isn't an equivalence of categories. Indeed, if $\varphi : G \to G'$ is a covering map, its differential at the identity $d_e\varphi : \mathfrak{g}' \to \mathfrak{g}$ is an isomorphism.

There are two ways of turning this functor into an equivalence of categories. Perhaps the simplest way is to restrict its domain to the full subcategory of *simply connected* Lie groups. But another way is to simply localize LieGrp with respect to all covering maps.[1] Then the universal property of localization gives an equivalence of categories between this localization and LieAlg. ■

A huge collection of examples are of the following form.

■ **Example 3.1.2 — Reflective localization.** Let D be a full subcategory of C. We say that D is a *reflective subcategory* if the inclusion functor $i : \mathsf{D} \to \mathsf{C}$ admits a left adjoint $r : \mathsf{C} \to \mathsf{D}$. Let S be the collection of morphisms in C which are sent to an isomorphism by r. Then D is equivalent to the localization $S^{-1}\mathsf{C}$. (Proposition 5.3.1 in [3].)

A plethora of examples of localization are of this form. The functor $\mathsf{Grp} \to \mathsf{Ab}$ sending a group to its abelianization identifies Ab as a localization of Grp. Similarly, the fraction field functor $\mathsf{IntDom} \to \mathsf{Fld}$, from the category of integral domains and injective morphisms to the category of fields, identifies Fld as a localization of IntDom. The reader which already has some knowledge of algebraic geometry may appreciate that both the sheafification functor and the functor

$$\begin{aligned} \mathsf{Sch} &\to \mathsf{Aff} \\ X &\mapsto \operatorname{Spec}\Gamma(X, \mathscr{O}_X), \end{aligned}$$

from the category of schemes to the category of affine schemes, are examples of reflective localization. ■

There are two issues with our notion of localization that ought to be addressed. Firstly, the localization of a locally small category need not be locally small.[2] This may be a problem for applying the Yoneda lemma, for example. Fortunately, almost all the localizations we are interested in will be locally small. (We'll soon see that the derived category of a locally small *Grothendieck* abelian category is locally small.)

Another problem with our notion of localization is that, if C is additive, it isn't clear if $S^{-1}\mathsf{C}$ is also additive or not. Indeed, how can we sum paths? We can solve this problem by forcing every path from M to N to be equivalent to a path of the form

$$\begin{tikzcd} & L & \\ M \arrow[ur, "f"] & & N, \arrow[ul, "s"'] \end{tikzcd}$$

---

[1] The reader may wonder if the composition of covering maps is still a covering map. Somewhat surprisingly, this is false in general, but it holds for manifolds due to the theorem 2.11 in [28].

[2] Or, using the formalism of Grothendieck universes, the localization of a category need not exist in our fixed universe.

which we call a *roof*. We'll then conclude that any two roofs can be written with the same morphism s on the right, allowing their sum.

Definition 3.1.1 — Multiplicative system. Let C be a category and S be a collection of morphisms in C. We say that S is a *left multiplicative system* if it satisfies:

(LMS1) S is stable under composition and contains all the identities of C.

(LMS2) For any pair of morphisms $f : L \to N$ in C and $s : L \to M$ in S, there exists $g : M \to L'$ in C and $t : N \to L'$ in S making the diagram

$$\begin{tikzcd}
& L' & \\
M \arrow[ur, dashed, "g"] & L \arrow[l, "s"'] \arrow[r, "f"] & N \arrow[ul, dashed, "t"']
\end{tikzcd}$$

commute.

(LMS3) For every pair of morphisms $f, g : L \to L'$ in C and $s : M \to L$ in S such that $f \circ s = g \circ s$, there exists $t : L' \to N$ in S such that $t \circ f = t \circ g$.

The conditions for a *right multiplicative* system are the same with all the arrows reversed. We say that S is a *multiplicative system* if it's both a right and a left multiplicative system.

While the axiom LMS3 may seem somewhat technical, the other two axioms are precisely what we need in order for every morphism in $S^{-1}\mathsf{C}$ to be represented by a roof. Indeed, if S is a left multiplicative system, the axiom LMS2 allows us to gather all the inverse arrows on the right side of the path and the axiom LMS1 says that all these inverse arrows become one single element of S.

Even better, we can detect equivalence of paths without ever leaving the realm of roofs. Formally, there exists an equivalence relation $\sim_L$ on roofs which induces a dashed isomorphism making the diagram

$$\begin{tikzcd}
\{\text{roofs from } M \text{ to } N\} \arrow[r, hook] \arrow[d, two heads] & \{\text{paths from } M \text{ to } N\} \arrow[d, two heads] \\
\{\text{roofs from } M \text{ to } N\}/\sim_L \arrow[r, dashed, "\sim"] & \mathrm{Hom}_{S^{-1}\mathsf{C}}(M, N)
\end{tikzcd}$$

commute. We say that two roofs

$$\begin{tikzcd}
& L_1 & & & & L_2 & \\
M \arrow[ur, "f_1"] & & N \arrow[ul, "s_1"'] & \text{and} & M \arrow[ur, "f_2"] & & N \arrow[ul, "s_2"']
\end{tikzcd}$$

are $\sim_L$ equivalent if there exists an object L in C and morphisms $p_1 : L_1 \to L, p_2 : L_2 \to L$ making the diagram

$$\begin{tikzcd}
 & & L & & \\
 & L_1 \arrow[ur, dashed, "p_1"] & & L_2 \arrow[ul, dashed, "p_2"'] & \\
M \arrow[ur, "f_1"] \arrow[urrr, "f_2"' near start] & & & & N \arrow[ul, "s_2"'] \arrow[ulll, "s_1" near start]
\end{tikzcd}$$

commute and such that $p_2 \circ s_2 = p_1 \circ s_1$ is in S. We observe that, if S is a right multiplicative system, every morphism in $S^{-1}C$ can be represented by a *trough*

$$\begin{tikzcd}
M & & N \\
 & L \arrow[ul, "s"] \arrow[ur, "f"'] &
\end{tikzcd}$$

and we have a similar relation $\sim_R$ for such diagrams. The next proposition proves all these claims. Since a right multiplicative system on C is nothing but a left multiplicative system on C^{op}, we'll henceforth only cite and prove results for left multiplicative systems, for analogous results hold by duality.

Proposition 3.1.3 Let S be a left multiplicative system in a category C, and let M, N be two objects of C. Then $\sim_L$ is an equivalence relation on the collection of roofs from M to N. Moreover, the canonical morphism sending a roof to a morphism in $S^{-1}C$ descends to the quotient defining an isomorphism

$$\mathrm{Hom}_{S^{-1}C}(M, N) \cong \{\text{roofs from } M \text{ to } N\}/\sim_L .$$

Proof. □

In precisely the same way that we sum fractions by writing them with a common denominator, we can write any two roofs with a single morphism s on the right.

Proposition 3.1.4 Let S be a left multiplicative system in a category C. Every two morphisms $M \to N$ in $S^{-1}C$ may be written as the equivalence classes of $s^{-1} \circ f$ and $s^{-1} \circ g$ for suitable morphisms f, g in C and $s \in S$.

Proof. □

Besides allowing the sum of two morphisms in a localization of a preadditive category, the above writing also allows us to easily decide whether two morphisms in the localization are equal. We claim that two morphisms in $S^{-1}\mathsf{C}$ represented by the roofs

$$\begin{tikzcd}
& L & \\
M \arrow[ur, "f_1"] & & N \arrow[ul, "s"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& L & \\
M \arrow[ur, "f_2"] & & N \arrow[ul, "s"']
\end{tikzcd}$$

are equal if and only if there exists a morphism $q : L \to L$ in C such that $q \circ s \in S$ and $q \circ f_1 = q \circ f_2$. Indeed, both roofs are equivalent if and only if there exist morphisms $p_1 : L \to L'$ and $p_2 : L \to L'$ making the diagram

$$\begin{tikzcd}
& & L' & & \\
& L \arrow[ur, dashed, "p_1"] & & L \arrow[ul, dashed, "p_2"'] & \\
M \arrow[ur, "f_1"] \arrow[urrr, "f_2"] & & & & N \arrow[ul, "s"'] \arrow[ulll, "s"]
\end{tikzcd}$$

commute and such that $p_2 \circ s = p_1 \circ s$ is in S. If there exists such a morphism q, we may take $p_1 = p_2 = q$. Conversely, the axiom LMS3 gives a morphism $t \in S$ such that $t \circ p_2 = t \circ p_1$ and we may take q to be this common morphism.

> **Corollary 3.1.5** Let S be a left multiplicative system in a preadditive category A. Then $S^{-1}\mathsf{A}$ is also preadditive, and the localization functor $Q : \mathsf{A} \to S^{-1}\mathsf{A}$ is additive. Moreover, if B is another preadditive category and $F : \mathsf{A} \to \mathsf{B}$ is an additive functor such that $F(s)$ is an isomorphism for every $s \in S$, the induced morphism $S^{-1}\mathsf{A} \to \mathsf{B}$ is also additive. If A is additive, then so is $S^{-1}\mathsf{A}$.

Proof. □

We're finally able to explain the *raison d'être* of the nomenclature and notation used in this section.

■ **Example 3.1.3 — Localization of noncommutative rings.** Let A be a (not necessarily commutative) ring. We define a category A which only has one object $*$ and such that $\mathrm{Hom}_{\mathsf{A}}(*, *) = A$. This is a preadditive category and, in this context, a left multiplicative system S on A is a subset of A such that

(a) S is multiplicatively closed and contains 1;

(b) for every $a \in A$ and $s \in S$, the set $As \cap Sa$ is nonempty;

(c) for every $a \in A$ and $s \in S$, if $as = 0$, then $ta = 0$ for some $t \in S$.

A particular case of the preceding corollary proves the existence of localizations for left multiplicative systems on any ring. This is a very important result on noncommutative ring theory. ■

For our next result, let C be a category and S be a left multiplicative system on C. Given an object N in C, we define a category N/S whose objects are morphisms $N \to L$ in S and whose morphisms are commutative diagrams

$$\begin{tikzcd} & N \arrow[dl] \arrow[dr] & \\ L_1 \arrow[rr] & & L_2, \end{tikzcd}$$

where the arrow $L_1 \to L_2$ is in C.

Corollary 3.1.6 Let S be a left multiplicative system in a category C. Then we may write $\operatorname{Hom}_{S^{-1}\mathsf{C}}(M, N)$ as the filtered colimit

$$\operatorname*{colim}_{(N \to L) \in N/S} \operatorname{Hom}_{\mathsf{C}}(M, L).$$

In particular, the localization functor $Q : \mathsf{C} \to S^{-1}\mathsf{C}$ commutes with finite colimits. Similarly, if S is a right multiplicative system, Q commutes with finite limits.

Proof. ☐

As with (pre)additive categories, a localization of an abelian category with respect to a multiplicative system is still abelian and satisfies a stronger universal property.

Proposition 3.1.7 Let S be a multiplicative system in an abelian category A. Then $S^{-1}\mathsf{A}$ is also abelian, and the localization functor $Q : \mathsf{A} \to S^{-1}\mathsf{A}$ is exact. Moreover, if B is another abelian category and $F : \mathsf{A} \to \mathsf{B}$ is an exact functor such that $F(s)$ is an isomorphism for every $s \in S$, the induced morphism $S^{-1}\mathsf{A} \to \mathsf{B}$ is also exact.

Proof. ☐

There's another point of view which is often used when dealing with localizations of abelian categories. For that we need the definition below.

Definition 3.1.2 — Thick subcategory. Let A be an abelian category. We say that a non-empty full subcategory B of A is *thick* if for any short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

in A, B is in B if and only if A and C are.

Due to the corollary 1.2.2, a thick subcategory is always abelian. The *raison d'être* of such subcategories is the result below.

Proposition 3.1.8 Let A be an abelian category. Given a multiplicative system S in A, the full subcategory B_S, composed of the objects which are isomorphic to 0 in $S^{-1}\mathsf{A}$, is thick. Conversely, given a thick subcategory B, the collection S_{B} of all morphisms φ in A such that $\ker \varphi$ and $\operatorname{coker} \varphi$ are in B is a multiplicative system.

Proof. □

Motivated by the proposition above, we define the *quotient* A/B of an abelian category A by a thick subcategory B as the localization $S_{\mathsf{B}}^{-1}\mathsf{A}$. These quotients are often called *Serre quotients* in the literature.

Given an exact functor $F : \mathsf{A} \to \mathsf{B}$ between abelian categories, its *kernel* is the full subcategory of A composed of the objects whose image by F is zero. It's clear that the kernel of an exact functor is thick. As in basically every algebraic category, the existence of quotients gives the converse. In this case, the last two propositions imply that every thick subcategory B of A is the kernel of some exact functor. Namely, the quotient / localization functor $Q : \mathsf{A} \to \mathsf{A}/\mathsf{B}$.

In due time, we'll see that many interesting abelian categories are Serre quotients of A-Mod, for some (not necessarily commutative) ring A. (Theorem **??**.) We present two other examples of Serre quotients.

■ **Example 3.1.4** Let S be a multiplicative subset of a ring A and B be the category of A-modules whose elements are annihilated by some element of S. It's clear that B is a thick subcategory of A-Mod. We affirm that $S^{-1}A$-Mod is canonically equivalent to A-Mod/B.

Let $f : A \to S^{-1}A$ (resp. $Q : A\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}/\mathsf{B}$) be the localization map (resp. functor). The functor $f^* : A\text{-}\mathsf{Mod} \to S^{-1}A\text{-}\mathsf{Mod}$, which sends M to $M \otimes_A S^{-1}A \cong S^{-1}M$, is exact and maps elements of B to zero. The universal property then implies that it descends to an exact functor $\tilde{f^*} : A\text{-}\mathsf{Mod}/\mathsf{B} \to S^{-1}A\text{-}\mathsf{Mod}$.

Denoting by $f_* : S^{-1}A\text{-}\mathsf{Mod} \to A\text{-}\mathsf{Mod}$ the restriction of scalars functor, the adjunction $f^* \dashv f_*$ gives rise to another adjunction $\tilde{f^*} \dashv Q \circ f_*$. The unit of the latter is a natural isomorphism, which proves that $\tilde{f^*}$ is fully faithful. Finally, it's also essentially surjective since the restriction of scalars of a $S^{-1}A$-module N to A is sent to an isomorphic copy of N. This finishes the proof.

In particular, the quotient of Ab by the thick subcategory of torsion groups is equivalent to $\mathbb{Q}$-Vect. ■

The reader that already knows some algebraic geometry may be pleased to know that the basic theory of quasicoherent sheaves on projective schemes may be phrased using Serre quotients.

■ **Example 3.1.5** Let A be a $\mathbb{N}$-graded ring, which is finitely generated by A_1 as an A_0-algebra, and $X = \operatorname{Proj} A$. We denote by A-GrMod the category of graded A-modules

M *such that* $\bigoplus_{d>n} M_d$ *is finite for some* n. The usual tilde functor

$$\begin{aligned} r : A\text{-}\mathsf{GrMod} &\to \mathsf{QCoh}(X) \\ M &\mapsto \widetilde{M} \end{aligned}$$

is exact and its kernel, denoted by $A\text{-}\mathsf{GrMod}_0$, is composed by the modules M satisfying $M_d = 0$ for all d large enough. [22, Proposition 2.7.3] The tilde functor r admits a right adjoint $\Gamma_\bullet$, defined by

$$\Gamma_\bullet(\mathscr{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n)),$$

which is fully faithful due to the fact that the counit

$$\widetilde{\Gamma_\bullet(\mathscr{F})} \to \mathscr{F}$$

is a natural isomorphism. Then, the formalism of example 3.1.2 implies that r factors through the quotient and that

$$A\text{-}\mathsf{GrMod}/A\text{-}\mathsf{GrMod}_0 \to \mathsf{QCoh}(X)$$

is an equivalence of categories. ∎

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